REAL ANALYSIS

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CHAPTER 1

ELEMENTARY SET THEORY AND COUNTABILITY

1.1. Definition

1.1.1. Set

A set is a well defined collection of distinct objects. By well defined we mean there is no confusion regarding inclusion or exclusion of objects. Sets are conventionally denoted with capital letters and small letters will represent the members of the set.

Note: This definition is widely accepted but not absolute.

Following notations will be used for some of the specific sets that are commonly used.

$\mathbb{N}$ = The set of all natural numbers or all positive integers.

$\mathbb{Z}$ = The set of all integers.

$\mathbb{Q}$= The set of all rational numbers.

$\mathbb{Q}^c$ = The set of all Irrational numbers.

Note: No set is member of itself.

i.e. $A \notin A$ for every set A.

Russell's Paradox: There is no set of all sets.

Reason: Let if possible, X be the set of all sets

i.e. $X = \{A \mid A \text{ is set}\}$

By definition of set, X is set.

And By definition of X

$X \in X$

Which is a contradiction.

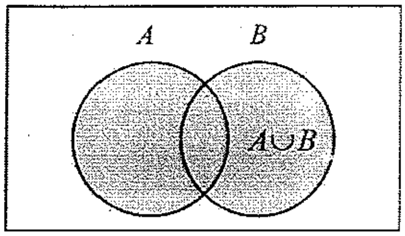
1.1.2. Operations on Sets

Let A and B be two sets. Then

1. **Union:** The union of A and B, denoted by $A \cup B$, is defined as the set $\{x \mid x \in A \text{ or } x \in B\}$.

   While writing $A \cup B$, elements common to A and B should be taken only once in the union.

Also note that $A \cup A = A$ and if $A \subset B$, then $A \cup B = B$



Example:

(a) Take $A = \{1,2,3,4\}$
$B = \{3,4,5,6\}$

Then $A \cup B = \{1,2,3,4,5,6\}$

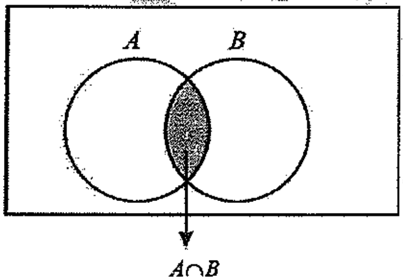
(b) If $A = \mathbb{Q}$
$B = \mathbb{Q}^c$

Then $A \cup B = \mathbb{R}$

2. **Intersection:** The intersection of A and B, denoted by $A \cap B$, is defined as the set $\{x \mid x \in A \text{ and } B \text{ both}\}$.

   Thus $A \cap B$ consists of all elements common to A and B. Also note that $A \cap A = A$ and if $A \subset B$ then $A \cap B = A$.

   If $A \cap B = \phi$, then the sets A and B are called **Disjoint**.



Example:

(a) Let $A = \{1,2,3,4\}$
$B = \{3,4,5,6\}$

Then $A \cap B = \{3,4\}$

(b) If $A = \left\{\frac{1}{n},\ n \in \mathbb{N}\right\}$ and $B = \left\{-\frac{1}{n},\ n \in \mathbb{N}\right\}$

Then $A \cap B = \phi$

3. **Difference:** The difference of A and B, denoted by $A - B$, is defined as the set $\{x \mid x \in A \text{ and } x \notin B\}$.

   Thus $A - B$ is the set of all elements of A not belonging to B.

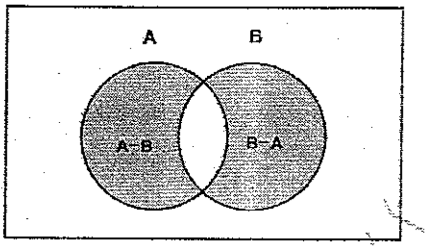
   Similarly, we can define $B - A$

Note:

(a) $A-B \neq B-A$

(b) $A-B \subset A, B-A \subset B$

(c) $(A-B) \cap (B-A) = \phi$



Example:

(a) If $A=\{1,2,3,4,5,6,7\}$ and $B=\{4,5,6,7,8\}$

Then $A-B=\{1,2,3\}$

(b) If $A=\mathbb{R}$
$B=\mathbb{Q}$

Then $A-B=\mathbb{R}-\mathbb{Q}=\mathbb{Q}^c$

4. **Complement:** The complement of A in the set U (universal set), denoted by A^c, is defined as the set $\{x \mid x \in U \text{ such that } x \notin A\}$. In other notation we can see it as $U - A$.

From this definition, it is easy to verify that $U = A \cup A^c, A \cap A^c = \phi, (A^c)^c = A$ and if $A, B \subset U$ and $A \subset B$ then $B^c \subset A^c$.

Example:

(a) If $A=\{1,2,3,4,5\}$ and

$U=\{1,2,3,4,5,6,7,8,9,10\}$

Then $A^c=\{6,7,8,9,10\}$

(b) If $A=\mathbb{Q}$
$U=\mathbb{R}$

Then $A^c=\mathbb{Q}^c$

5. **Power Set:** The power set of A, denoted by $P(A)$, is defined as a set $\{X \mid X \subseteq A\}$. Thus, $P(A)$ is collection of all possible subsets of A.

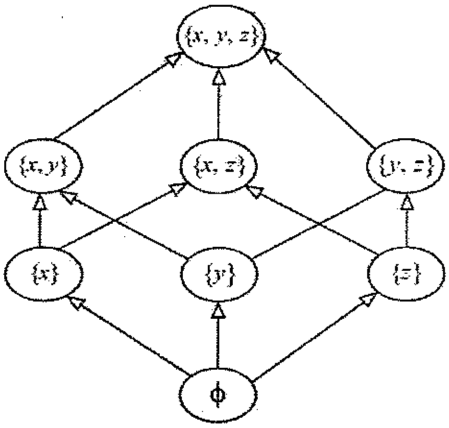
If $|A|=n$, Then $|P(A)|=2^n$

(where, $|A|$ denotes the number of elements in the set A. i.e, cardinality of the set A)

If $A=\{x,y,z\}$

Then

$P(A)=\{\phi,\{x\},\{y\},\{z\},\{x,y\},\{x,z\},\{y,z\},\{x,y,z\}\}$



The cardinality of the set $\{x,y,z\}$, is three, while there are eight elements in its power set.

6. **Cartesian Product:** If A and B are nonempty sets, then the Cartesian product $A\times B$ of A and B is the set of all ordered pairs (a,b) with $a\in A$ and $b\in B$. That is,

   $A\times B:=\{(a,b):a\in A,b\in B\}$.

   Thus if $A=\{1,2,3\}$ and $B=\{1,5\}$, then the set $A\times B$ is the set whose elements are the ordered pairs

   (1,1), (1,5), (2,1), (2,5), (3,1), (3,5)

   We may visualize the set $A\times B$ as the set of six points in the plane with the coordinates that we have just listed.

   Note: $A\times B\neq B\times A$ when $A\neq B$ and also if any of A and B is empty then $A\times B=\phi$.

   Relation: Any subset of $A\times B$ defines a relation from A to B.

   Example: Let $A=\{1,2,3\}$ and $B=\{4,5\}$ define

   $R=\{(x,y):|x-y| \text{ is even}\}$

   $=\{(1,5),(2,4),(3,5)\}$

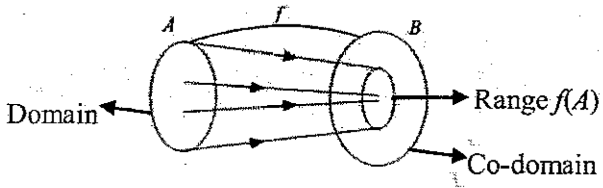
1.1.3. Functions And Their Properties

1. **Function on a set:** A relation f from a non-empty set A to another non empty set B is said to be a function, if each element $x\in A$ corresponds to a unique element $y\in B$ and the y which corresponds in this way to a given x is denoted by $f(x)$, and is called the value of x under f.

   Note: If f is a function from a set A to a set B then we write it as $f:A\to B$.

2. **Graph:** If f is a function from A to B, then set of all ordered pairs $(x,f(x))$ is called the graph of the function f.

3. **Domain, Co-domain & Range of a Function:** For a function f from A to B, the set A is called the domain of f and the set of values of f is called the range of f and written as $f(A)$, whereas B is called the Co-domain of f. Clearly $f(A) \subseteq B$. All these sets can be seen in following figure easily.



Equality of Two Functions: Two functions f and g are said to be equal if

(a) Domain of f = Domain of g

(b) $\forall x, f(x) = g(x)$

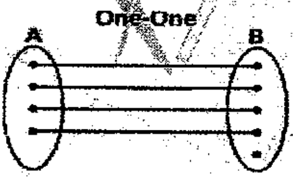
Note: The terms 'function' and 'mapping' are synonyms.

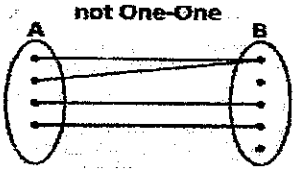
4. **Counting of Functions:** Let A and B be two finite sets having m and n elements respectively. Then each element of A can be associated to any one of n elements of set B. So, total number of functions from set A to set B is equal to ways of having m job where each job can be done in n ways. Hence total number of function from A to B is n^m or $|B|^{|A|}$

Example: Let A be the set containing 10 elements and B be the set containing 5 elements. Then total number of functions from A to B is 5^{10}.

1.1.4. Classification of Functions

1. **One-One (Injective) Function:** We say a function $f : A \rightarrow B$ is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \; \forall x_1, x_2 \in A$ i.e. no two elements in the domain correspond to the same element in the range.





Algorithm to Check Injectivity of Function

Step I: Take two arbitrary elements x_1 and x_2 in the domain

Step II: Put $f(x_1) = f(x_2)$

Step III: Solve $f(x_1)=f(x_2)$. If $f(x_1)=f(x_2)$ gives $x_1=x_2$ only, then $f:A\to B$ is a one – one function otherwise not.

Example: $f:[0,1]\to\mathbb{R}$ defined by

$f(x)=x^2$ is a one-one function.

$\because\ f(x)=f(y)$

$\Rightarrow\ x^2=y^2$

$\Rightarrow\ x=y \qquad (\because\ x,y\geq 0)\ \forall\, x,y\in[0,1]$

Remark:

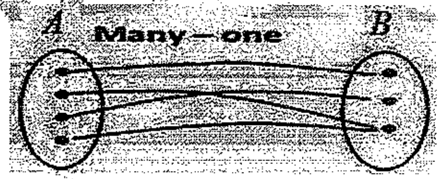
(a) Injectivity of a function can also be checked from its graph. If any straight line parallel to x-axis intersects the curve $y=f(x)$ at most at one point, then the function $f(x)$ is one-one or an injection otherwise it is not.

(b) If $f:A\to B$ then $x_1=x_2\Rightarrow f(x_1)=f(x_2)$ is true for all $x_1,x_2\in A$. However $f(x_1)=f(x_2)\Rightarrow x_1=x_2$ for all $x_1,x_2\in A$ is true only when f is an injective map.

(c) If A and B are finite sets having m and n elements respectively then number of one-one functions from A to $B=\begin{cases}{}^nP_m, & \text{if } n\geq m\\ 0 & \text{if } n<m\end{cases}$ where

$${}^nP_m=\frac{n!}{(n-m)!}$$

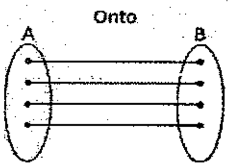
2. **Many-One Function:** A function $f:A\to B$ is said to be many-one function if two or more elements of set A (if exists) have the same image in B.

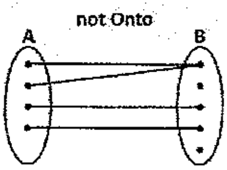


Example: Consider the function $f[-1,1]\to\mathbb{R}$, defined by $f(x)=x^2$, then f is many one function.

3. **Onto (Surjective) Function:** A function $f:A\to B$ is said to be an onto function or a surjective function if every element of B is the f-

image of some element of A i.e. if $f(A)=B$ or range of f is the co-domain of f.





Remark: It depends upon domain and co-domain respectively for a function to be one-one and onto.

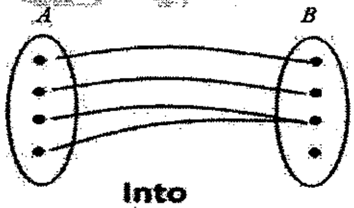
Example: The function $f:[0,1]\to\mathbb{R}$ defined by $f(x)=x^2$ is a one-one function where as function $f:[-1,1]\to\mathbb{R}$ defined by $f(x)=x^2$ is not a one-one function. ($\because\ f(-1)=f(1)=1$)

Also function $f:[0,1]\to\mathbb{R}$ defined by $f(x)=x^2$ is not an onto function, but $f:[0,1]\to[0,1]$ defined by $f(x)=x^2$ is an onto function.

Remark: If A and B are two sets having m and n elements respectively and let $1\le n\le m$, then number of onto functions from A to B is $\left[n^m - \left[\sum_{r=1}^{n}(-1)^{r+1}\,n_{c_r}\,(n-r)^m\right]\right]$ where $n_{c_r}=\dfrac{n!}{r!(n-r)!}$.

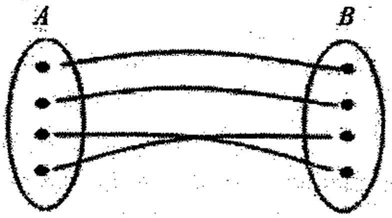
Example: If $|A|=4, |B|=3$ then there are 36 onto maps from A to B.

4. **Into Function:** A function of $f:A\to B$ is an into function if there exists an element in B having no pre-image in A. In other words, $f:A\to B$ is an into function if $f(A)\subset B$.



Into

5. **One–One Onto (Bijective) Function:** The map $f:A\to B$ is bijective if it is one-one as well as onto.



Bijection (one- one onto)

Example: The map $f:\mathbb{N}\to\mathbb{Z}$ defined as $f(n)=(-1)^n\left[\frac{n}{2}\right]$

(Where $\left[\frac{n}{2}\right]$ denotes the greatest integer function) is bijective function

Cantor's Theorem: For any set A, there is no map from A onto the set $P(A)$, the power set of A.

Reason: Let $\phi:A\to P(A)$ is onto.

Since for every $x\in A$, $\phi(x)$ is a subset of A. Then either $x\in\phi(x)$ or it does not. Now consider $X=\{x\in A : x\notin\phi(x)\}$

Since X is a subset of A and ϕ is onto, then $X=\phi(t)$ for some $t\in A$, so either $t\in X$ or $t\notin X$. If $t\in X$ and since $X=\phi(t)$. We must have $t\in\phi(t)$. Which is contrary to the definition of X.

Similarly, If $t\notin X$ then $t\notin\phi(t)$ so that $t\in X$ which is also a contradiction.

So ϕ cannot be onto.

Remark: There always exists a one-one map from A to $P(A)$.

1.2. Countable and Uncountable Sets

1. **Similar/Equivalent/Equipotential Sets:** Two set A and B are called Similar if and only if there exists an bijective function between them and then we write $A\sim B$
2. **Finite Set:** Consider $S^{(n)}=\{1,2,3,4,\ldots,n\}$ .Then a non-empty set A is said to be finite iff there exist $n\in\mathbb{N}$ and a bijective function f from $S^{(n)}$ to A. (n gives the number of elements in the set A)

   If no such n exist, then set A is said to be infinite.

   Example: Set of all vowels $A=\{a,e,i,o,u\}$ is a finite set, since there exist $f:A\to S^{(5)}$ which is a bijection.

   Note:

   (a) Two finite sets are similar if their cardinality are equal.

   (b) Every set A is equivalent to itself as there exists an identity map on A.

   (c) If $A\sim B$ and $B\sim C$ then $A\sim C$.

   (d) The main difference between a finite and an infinite set is that every infinite set is equivalent to at least one of its proper subset whereas a finite set can never be equivalent to any of its proper subset.

   #It is so important that some of the authors use this note as the definition of finite and infinite sets.

   (e) Subset of a finite set is a finite set.

   (f) Empty set is considered as a finite set.

3. **Countably Infinite Set:** A set S is said to be countably infinite if it is similar to set of natural numbers i.e. if $S \sim \mathbb{N}$.

   Example: $\mathbb{Z}$ is a countably infinite set.

   Reason: Define $\phi : \mathbb{N} \to \mathbb{Z}$ as

   $$\phi(n) = (-1)^n \left[\frac{n}{2}\right]$$

   Clearly ϕ is bijective function from $\mathbb{N}$ to $\mathbb{Z}$

   $\therefore$ $\mathbb{Z}$ is a countably infinite set.

   In Abstract: We say a set can be put under counting system or process if a rule can be imposed over that set by which successor is defined.

4. **Countable Set:** A set S is called countable if it is either finite or countably infinite. In case of countably infinite there is function f which establishes a bijection between $\mathbb{N}$ and the elements of S . Therefore the set S can be displayed as $S = \{f(1), f(2), ..., f(n), ...\}$. If we write $\{a_1, a_2, ..., a_n, ...\}$ in place of $\{f(1), f(2), ..., f(n), ...\}$ then $S = \{a_1, a_2, ..., a_n, ...\}$

5. **Uncountable Set:** A set which is not a countable set, called uncountable set.

   Note: If every map from A to B fails to be onto. Then we say B is larger than A in the sense of cardinality, i.e. the co-domain has more potential than the domain.

   Or,

   If the domain has less potential then every map from A to B fails to be onto.

6. Let A and B are finite set and $|A| < |B|$. Then A is similar to a subset of B and B is not similar to any of the subset of A.

   Hence $|A| = |B|$ iff $A \sim X \subseteq B$ and $B \sim Y \subseteq A$

   Note: If we define cardinality of infinite sets no matter how it is defined but these cardinality will follow the above rules.

   Cardinality of infinite set: For infinite sets we use symbols to denote their cardinality

   $|\mathbb{N}| = \aleph_0$ **(Aleph naught)**

   $|\mathbb{R}| = c$ **(Continum)**

   These symbols which are used to denote cardinality of infinite sets are known as transfinite numbers.

7. **Arithmetic of Cardinal Numbers:** $\aleph_0$ denotes the cardinality of $\mathbb{N}$ and c denotes the cardinality of $\mathbb{R}$.

   (a) $\underbrace{\aleph_0 + \aleph_0 + ... + \aleph_0}_{n \text{ times}} = \aleph_0$

   (b) $n + \aleph_0 = \aleph_0, n \in \mathbb{N}$

   (c) $\aleph_0 + c = c$

(d) $\underbrace{c+c+....+c}_{n \text{ times}}=c$

(e) $n \cdot \aleph_0 = \aleph_0, n \in \mathbb{N}$

(f) $c \cdot c \cdot c \cdot ..... \cdot c = c$

(g) $\aleph_0 \cdot c = c$

(h) $2^{\aleph_0} = c > \aleph_0$

(i) $\aleph_0^{\aleph_0} = 2^{\aleph_0} = c$

(j) $\aleph_0^c = 2^c$

(k) $\aleph_0 < c$

(l) $n < \aleph_0$

For any transfinite number α, β, γ

(m) $\alpha+\beta=$ larger of α and β

(n) $\alpha\beta=$ larger of α and β

(o) $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$

(p) $\left(\alpha^\beta\right)^\gamma = (\alpha)^{\beta\gamma}$

(q) $(\alpha\beta)^\gamma = \alpha^\gamma \cdot \beta^\gamma$

(r) $\alpha^\gamma \le \beta^\gamma$ if $\alpha \le \beta$

(s) $\gamma^\alpha \le \gamma^\beta$ if $\alpha \le \beta$

8. There is no transfinite number between $\aleph_0$ and c.
9. If A and B are two sets with cardinality α and β respectively and $F = \{f \mid f : A \to B\}$ then card $|F| = \beta^\alpha$
10. $F_1 = \{f \mid f : \mathbb{N} \to \mathbb{N}\}$, then card $|F_1| = \aleph_0^{\aleph_0} = 2^{\aleph_0} = c \Rightarrow F_1$ is uncountable.
11. $F_2 = \{f \mid f : [0,1] \to [0,1]\}$ Then card $|F_2| = c^c = 2^c \Rightarrow F_2$ is uncountable
12. The set of all equivalence classes of an equivalence relation on a set is called Quotient set.

    (a) Quotient set of countable set is always countable.

    (b) Quotient set of uncountable set may be countable or uncountable.

1.3. Properties of Sets

Put Your Own Notes

1. Two finite set are similar iff they have same number of elements.
2. Every countable set is equivalent to a subset of natural numbers.

   Reason: Let A be any countable set. Now if A is finite then it is similar to $S^{(n)}$ for some n.

   Else if A is countably infinite then A can be written as

   $A = \{a_1, a_2, a_3, ...\}$

   Now define $\phi : A \to \mathbb{N}$ as $\phi(a_k) = k \quad \forall\ k \in \mathbb{N}$

   This ϕ is a bijective map and thus A is equivalent to $\mathbb{N}$.
3. Every set equivalent to finite/infinite set is finite/infinite set.

   Reader should prove it themselves.
4. If A is similar to a subset of B and B is similar to subset of A, then $A \sim B$

   Reason: As A is similar to a subset of B

   $\Rightarrow$ A has less potential than B ...(1)

   Also B is similar to a subset of A

   $\Rightarrow$ B has less potential than A ...(2)

   By (1) and (2)

   $A \sim B$
5. A set is infinite iff it contains a countabily infinite subset.

   Reason: Let X be any infinite set.

   Take $x \in X$ and name it as a_1 and define $A_1 = \{a_1\}$

   Again take another $y \in X - A_1$ and name it as a_2

   Define $A_2 = \{a_2\} \cup A_1$

   $\vdots$

   Proceeding in same way, we have

   $A_n = A_{n-1} \cup \{a_n\} \qquad \forall\ n \in \mathbb{N}$

   Let $A = \bigcup_{n \in \mathbb{N}} A_n = A_1 \cup A_2 \cup ... = \{a_1, a_2, ... a_n, a_{n+1}, ...\}$

   Clearly $A \subset X$

   This shows that every infinite set has countable infinite subset

   Converse is obvious.

   Hence a set X is infinite iff it contains a countably infinite subset.
6. Every infinite set is equivalent to at least one of its proper subset.

Reason: As done in above property construct $A = \bigcup_{n \in \mathbb{N}} A_n$

Now take $B = A - \{a_1\}$ and define $Y = (X - A) \cup B$

Clearly $a_1 \notin Y \subset X$ thus Y is proper subset of X

Define $g : X \to Y$ such that

$$g(x) = \begin{cases} a_{i+1} & x = a_i \\ x & \text{else} \end{cases}$$

Clearly g is one-one and onto.

This shows that if we remove one element from an infinite set. Then its cardinality does not change.

By the same procedure if we remove finite number of elements from an infinite set then its cardinality remains same.

7. A finite set is not equivalent to any of its proper subset.

   Reason: Let A is any finite set and $B \subset A$ then clearly B has less number of elements then A and two finite set are equivalent iff they have same number of elements so A and B can-not be equivalent. Thus no finite set can be equivalent to its proper subset.

8. Countable union of countable set is countable.

9. If B is an uncountable set and A is a countable set then $(B - A)$ is an uncountable set.

   Reason: Suppose that $(B - A)$ is countable set and also $B = (B - A) \cup A$ (being a union of two countable set) is countable thus B is a countable set. But this is not possible as B is an uncountable set. Thus $B - A$ is uncountable set.

10. Family of all finite subsets of countable set is countable.

    Reason: Let A be any countable set.

    If A is a finite set then nothing to prove. Now, Suppose A is countably infinite set.

    Then $A \approx \mathbb{N}$ If we show that collection of all finite subsets of $\mathbb{N}$ is countable then we are done.

    Now define A_1 = union of all subsets of $\mathbb{N}$ having single element i.e. $A_1 = \{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup ...$

    Thus, A_1 is countable.

    Now, Let A_2 = Union of all subset of $\mathbb{N}$ having 2 elements. Then A_2 is countable.

    In this manner A_n is countable $\forall n \in \mathbb{N}$

    Thus if $\mathcal{F}$ denotes the collection of all finite subset of $\mathbb{N}$ then $\mathcal{F}$ $= \bigcup_{n=1}^{\infty} A_n$ which is countable (being countable union of countable sets).

11. The family of all infinite subsets of an infinite set X is uncountable.

Reason: Let $\mathscr{B}$ denote the family of all subsets of the set X and $\mathscr{A}$ denote the family of all finite subsets of X. Then family of all infinite subset of the set X is equal to $\mathscr{B}-\mathscr{A}$. Which is uncountable.

(By Property 9)

Remark:

(a) The family of all subsets of countably infinite set is uncountable.

(b) $P(\mathbb{N}), P(Q), P(\mathbb{Z})$ all are uncountable set.

12. Finite Cartesian product of countable set is countable.

Application: $\mathbb{N}\times\mathbb{N}\times\ldots\times\mathbb{N}$ is countable set.

13. The set of all positive rational numbers is countable.

Reason: $\mathbb{Q}^+=\left\{\frac{p}{q}; p,q \text{ are co-prime positive integers}\right\}$

Let $A=\{(p,q); p,q \text{ are co-prime positive integers}\}$

Clearly the elements of $\mathbb{Q}^+$ and A are in one-one correspondence and therefore $\mathbb{Q}^+$ is countable iff A is countable. Since $A\subset\mathbb{N}\times\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$ is countable, therefore, A is countable (being subset of countable set)

Hence $\mathbb{Q}^-$ is also countable

Note:

(a) $\mathbb{Q}=\mathbb{Q}^+\cup\mathbb{Q}^-\cup\{0\}$ is countable

(b) By view of 12 & 13, we can say $\mathbb{Z}\times\mathbb{Z}\times\ldots\times\mathbb{Z}$, $\mathbb{Q}\times\mathbb{Q}\times\ldots\times\mathbb{Q}$ all are countable sets.

14. Infinite Cartesian product of countable sets $A_1, A_2, A_3, \ldots A_n\ldots$ (where $|A_i|\geq 2$, for infinitely many values of i) is uncountable.

15. Set of all polynomial with rational coefficients is a countable set.

Reason: Let $P_m=\left\{a_0+a_1x+\ldots+a_mx^m : a_i\in\mathbb{Q}\right\}$

Then P_m is countable $\forall m\in\mathbb{N}\cup\{0\}$ as there is a bijection $\phi: P_m\to\mathbb{Q}\times\mathbb{Q}\times\ldots\times\mathbb{Q}$ defined as

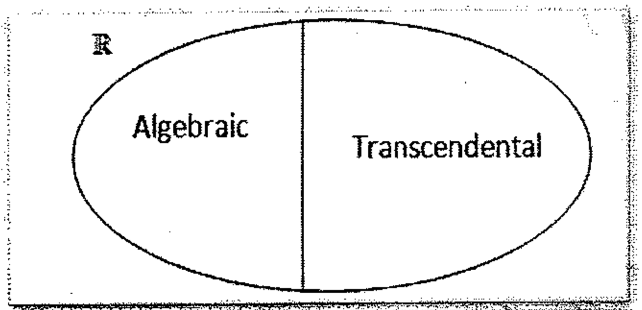
$$\phi\left(a_0+a_1x+\ldots+a_mx^m\right)=\left(a_0,a_1,\ldots,a_m\right)$$

Now if P denote the class of all polynomials over $\mathbb{Q}$ then clearly $P=\bigcup_{n=1}^{\infty}P_n$ which (being countable union of countable set) is countable.

16. Set of all circles whose Centre and radius are rational number is countable.

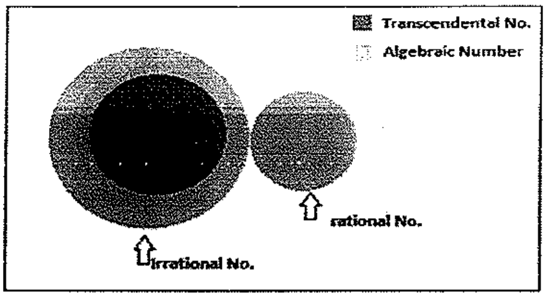
Reason: Let $X=\{c \mid c \equiv (x-a)^2+(y-b)^2=r^2 \text{ where } a,b,r\in\mathbb{Q}\}$

Define $\phi : X \to \mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$ as $\phi(c)=(a,b,r)$. This ϕ is a bijective map and thus X a countable set.



Algebraic number: A real number is called an algebraic number, if it is a root of a polynomial $P(x)=a_0+a_1x+\ldots+a_nx^n=0 \ (a_n\neq 0)$ with rational coefficient.

Transcendental number: A real number which is not **Algebraic** is called a transcendental number.



17. Set of all algebraic numbers and set of all transcendental numbers forms a partitions of the set $\mathbb{R}$.
18. Every rational number is algebraic.
19. Every transcendental number is irrational.
20. Irrational number may be algebraic or transcendental.

    Example:

    $\sqrt{2}$ is an algebraic number and π is a transcendental number.
21. Set of all algebraic number is countable.

    Reason: For a fixed $m\in\mathbb{N}\cup\{0\}$ define $P_m=\left\{a_0+a_1x+a_2x^2+\ldots+a_mx^m \mid a_i\in\mathbb{Q}\right\}$. This P_m is countable.

    Now take $P=\bigcup_{m\in\mathbb{N}} P_m$ and being countable union of countable sets it is also countable. So, P is countable.

    Now if α is any algebraic number then there is a polynomial in P such that whose root is α.

Thus the set of all algebraic numbers is contained in the set of zeros of polynomials in P. Now all we need to show is that the set of zeros of these polynomials is countable.

Since every polynomial can have at most k real roots where k is degree of that polynomial. Now we denote the set A_m as the set of zeros of polynomials in P_m. Since P_m has countable number of polynomial and further all of those polynomial have finite number of zeros so A_m will have countable number of elements.

Now define $A = \bigcup_{m\in\mathbb{N}} A_m$, the set of zeros of polynomials in P, being countable union of countable set is countable and set of algebraic numbers being subset of this set.

$\therefore$ Set of algebraic numbers is countable.

22. Set of all transcendental numbers is uncountable.

Reason: Since $\mathbb{R} = \mathcal{A} \cup \mathcal{T}$ where $\mathcal{A}$ is set of all algebraic numbers and $\mathcal{T}$ is set of all transcendental numbers. Now since $\mathcal{A}$ is countable and $\mathbb{R}$ is uncountable

$\therefore \mathcal{T} = \mathbb{R} - \mathcal{A}$ Is uncountable **(By using the result 9)**

23. Arbitrary collection of disjoint open intervals is countable.

Reason: consider $\mathcal{I}$ to be the collection of disjoint open intervals. Now if I is any interval then we can find a rational number which is not in any other interval we denote that by a_1 if we take any other interval there we can find any other rational number a_2 in this way we can find a sequence of rational numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ where each a_i is from different interval and thus we can generate a set of rational numbers having one–one correspondence with the set $\mathcal{I}$ and since that set of rational number is countable so $\mathcal{I}$ is also countable.

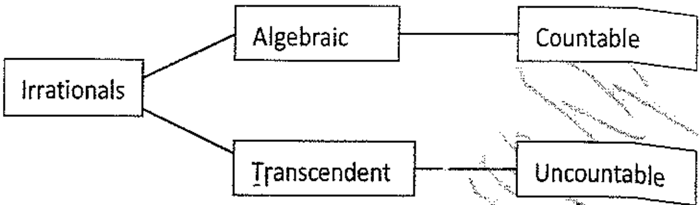
24. There does not exist any onto function from a countable set to an uncountable set.

25. There does not exist any one-one function from an uncountable set to a countable set.

26. **Some special Sets**

Sets	Countable / Uncountable
Natural Numbers ($\mathbb{N}$)	Countable
Integers ($\mathbb{Z}$)	Countable
Rational Numbers ($\mathbb{Q}$)	Countable
Non Empty , non-singleton Intervals (Open, closed, semi-open, semi-closed)	Uncountable

Irrationals $\left(\mathbb{Q}^c\right)$	Uncountable
Real numbers $(\mathbb{R})$	Uncountable
Algebraic numbers $(\mathcal{A})$	Countable
Transcendental Numbers $(\mathcal{T})$	Uncountable



27. There exist intervals which are countable e.g. $[a,a]$ $\forall a \in \mathbb{R}$
28. Every countably infinite set has cardinality $\aleph_0$ **(Aleph Naught)**
29. 'c' is the smallest cardinality of any uncountable set.
30. The unit interval $[0,1]$ is uncountable set

Reason: Let us assume that $[0,1]$ is countable.

$\Rightarrow$ either $[0,1]$ is finite or countably infinite .

Since every non empty, non singleton interval is an infinite set, $[0,1]$ is Countably infinite.

$\Rightarrow$ There is an enumeration $x_1, x_2, x_3, \ldots$ of real numbers in $[0,1]$.

Expanding each x_i in the form of an infinite decimal, we have

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}a_{14}\ldots a_{1n}\ldots \\ x_2 &= 0.a_{21}a_{22}a_{23}a_{24}\ldots a_{2n}\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= 0.a_{n1}a_{n2}a_{n3}a_{n4}\ldots a_{nn}\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

where each $a_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}$.

Consider the number b with decimal representation $b = 0.b_1 b_2 b_3 \ldots b_n \ldots$

where b_1 is any integer from 0 to 9 such that $b_1 \neq a_{11}$

b_2 is any integer from 0 to 9 such that $b_2 \neq a_{22}$

……………………………………………………

b_n is any integer from 0 to 9 such that $b_n \neq a_{nn}$ and so on.

Clearly, $b \in [0,1]$ and $b \neq x_n \ \forall n$ since the decimal representation of b is different from the decimal representation of x_n as $b_n \neq a_{nn}$. Thus b escapes enumeration and we arrive at a contradiction.

Hence $[0,1]$ is not countable.

31. $(0,1)$ is uncountable set. **(By property 6)**

32. Any open interval (a,b) is uncountable where $a \neq b$

33. Any open interval (a,b) is equivalent to any other open interval (c,d) where $a \neq b$ & $c \neq d$

    Reason: Let $x \in (a,b)$. Consider a function $f:(a,b) \to (c,d)$ given by

    $$f(x) = c + \frac{d-c}{b-a}(x-a).$$

    It is easy to verify that f is one-one and onto. Hence the result follows.

34. Any closed interval $[a,b]$ is equivalent to any other closed interval $[c,d]$ where $a \neq b$ & $c \neq d$

35. Any two non-trivial intervals are equivalent (non-empty, non-singleton intervals are non-trivial)

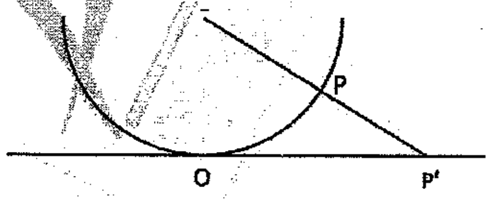
36. The set $\mathbb{R}$ of all real numbers is uncountable.

    Reason: To prove this, it is sufficient to show that $(0,1) \sim \mathbb{R}$. Consider the function $f:(0,1) \to \mathbb{R}$ given by

    $$f(x) = \begin{cases} \dfrac{2x-1}{x} & \text{if } 0 < x < \dfrac{1}{2} \\ \dfrac{2x-1}{1-x} & \text{if } \dfrac{1}{2} \leq x < 1 \end{cases}$$

    Clearly, f define a one-to-one correspondence between $(0,1)$ and $\mathbb{R}$. This proves that $(0,1) \sim \mathbb{R}$.

    Geometrical Reason: Let (a,b) be an open interval. We bend (a,b) into a semi-circle and rest this semi-circle tangentially on the real line $\mathbb{R}$ as shown in figure.



Let P be a point of (a,b). Then, by joining P with the centre of the semi-circle and producing it to meet the real line at P', we can locate a point on the real line corresponding to each point of (a,b); and conversely. This establishes a one-to-one correspondence between the sets (a,b) and $\mathbb{R}$. Hence $\mathbb{R}$ is uncountable.

37. Any uncountable subset of $\mathbb{R}$ is similar to $\mathbb{R}$.

38. $\mathbb{R} \sim (a,\ b)$ Or $[a,\ b]$ or $(a,\ b]$ or $[a,\ b)$ Also $\mathbb{R} \sim \mathbb{R}^n$ Implies $\mathbb{R} \sim \mathbb{C} \sim \mathbb{R}^2 \Rightarrow$ Set of all points of the plane is uncountable where $a \neq b$. Also $(0,\ 1) \times (0,\ 1) \sim (0,\ 1)$.

39. Let A be any set with $\text{card} A = n$ then $|P(A)| = 2^n \Rightarrow |P(\mathbb{N})| = 2^{\aleph_0} = c$, $|P(\mathbb{R})| = 2^c$.

40. Set of all functions having domain A (countably infinite) to the co-domain B (containing at least two points) is uncountable set.

1.4. Cantor Set

For describing the Cantor set, we shall give the following preliminaries on the ternary expansions of the real numbers in $[0,1]$ which is needed for the arithmetical characterization of the Cantor set.

In the ternary expansion of a real number x, we use the digits $0, 1 \text{ and } 2$. Thus in the ternary scale, $x = 0.a_1a_2a_3 \ldots$ means that $x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \ldots$ where $a_1, a_2, a_3,\ \ldots$, will take any one of the values $0, 1 \text{ and } 2$. Except for numbers like $\frac{1}{3}, \frac{2}{3}$, etc. The ternary expansion of every real number is unique.

In the case of $\frac{1}{3}$, we have the following two expansions:

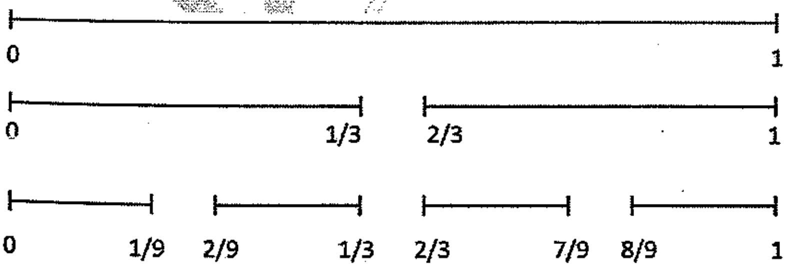
$\frac{1}{3} = 0.1000 \ldots,\ \frac{1}{3} = 0.0222 \ldots$

In the case of $\frac{2}{3}$, we have the following two expansions

$\frac{2}{3} = 0.1222 \ldots$ and $\frac{2}{3} = 0.2000 \ldots$

The numbers $\frac{1}{2}$ and $\frac{1}{4}$ have the following expansion,

$\frac{1}{2} = 0.11111 \ldots$ and $\frac{1}{4} = 0.020202 \ldots$



Now consider the closed interval $[0,1]$. From $[0,1]$, let us remove $\left(\frac{1}{3}, \frac{2}{3}\right)$ which the open middle third part is of $[0,1]$. Then we obtain the intervals

$\left[0,\frac{1}{3}\right]$ and $\left[\frac{2}{3},1\right]$. Now we shall remove the open middle thirds of each one of these intervals. That is we take away $\left(\frac{1}{9},\frac{2}{9}\right)$ in $\left[0,\frac{1}{3}\right]$ and $\left(\frac{7}{9},\frac{8}{9}\right)$ in $\left[\frac{2}{3},1\right]$ and get these intervals. $\left[0,\frac{1}{9}\right],\left[\frac{2}{9},\frac{1}{3}\right],\left[\frac{2}{3},\frac{7}{9}\right]$, and $\left[\frac{8}{9},1\right]$.

At each one of the succeeding stage, we shall remove the open middle third of each open of the remaining close intervals. If this process is carried for a countable number of times, removing the middle thirds of the left out intervals at each stage, the set of points left out in $[0,1]$ is called the Cantor ternary set or Cantor middle third.

First note that any number in $\left(\frac{1}{3},\frac{2}{3}\right)$ removed in the first stage has the expansion of the form $0.1a_2a_3a_4\dots$. For all numbers removed in the second state in $\left[\frac{1}{9},\frac{2}{9}\right]$ and $\left[\frac{2}{3},\frac{7}{9}\right]$, 1 will occur in the second place in the ternary scale on the assumption $\frac{2}{3}$ can be represented as $.011111\dots$. In a similar manner, all the numbers removed in the nth stage will have 1 in the nth position in the ternary scale. All the left out numbers will not have one in their expansions. Hence the Cantor set K is the set of all numbers x in $[0,1]$ which have a ternary expansion without digit 1. Where we assume that if the end points like $\frac{1}{3}$ and $\frac{2}{3}$ have two expansions, we shall take that expansion without one's.

Note: From the construction, it may appear that only the end points. $\frac{1}{3},\frac{2}{3},\frac{1}{9},\frac{2}{9},\frac{7}{9},\frac{8}{9},\dots$ are left alone in the Cantor set. But this is not so. For example $0.0202020\dots$ in the scale of 3 is $\frac{1}{4}$ and it is in the Cantor set, since 1 does not appear in the expansion.

Further it is interesting to note the sum of the lengths of the intervals removed in $[0,1]$ is one as seen below.

$$\frac{1}{3}+\left(\frac{1}{9}+\frac{1}{9}\right)+\left(\frac{1}{27}+\frac{1}{27}+\frac{1}{27}+\frac{1}{27}\right)+\dots=\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\dots=\sum_{n=0}^{\infty}\frac{1}{3}\cdot\left(\frac{2}{3}\right)^n$$

The sum of this geometric series is $\frac{1}{3}\cdot\frac{1}{1-\frac{2}{3}}=1$

Results on Cantor's Set:

1. $A=\left\{x \mid x=\sum_{n=1}^{\infty}\frac{a_n}{3^n}, a_n = 0 \text{ or } 2\right\}$ is another expression of Cantor set.
2. Cantor set contains no interior point.
3. Cantor set doesn't contain any open interval.
4. It is closed and bounded.
5. Each point of cantor set A is limit point of A.
6. Cantor set is perfect.
7. Measure of Cantor set is zero
8. Cantor set is uncountable.

Note: For definition of interior point, limit point, perfect set, closed and bounded set See Chapter 2.

CHAPTER 2

POINT SET TOPOLOGY

2.1. Definitions and Properties

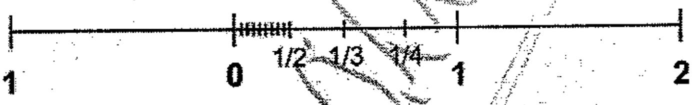
I Archimedean Property Of Real Numbers

1. If x, y are two positive real numbers then exist a positive integer n such that $ny > x$.
2. If x is any positive real number then there exist a positive integer n such that $n > x$.
3. For any real number x, there exist an integer n such that $\frac{1}{n} < x$

II Important Definitions

1. **Upper bound:** Let S be any subset of $\mathbb{R}$ then a real number M is called an **upper bound** of S if every element of S is less than or equal to M i.e. $x \le M \quad \forall x \in S$.

   Example: $S = \left\{\frac{1}{n}; n \in \mathbb{N}\right\}$

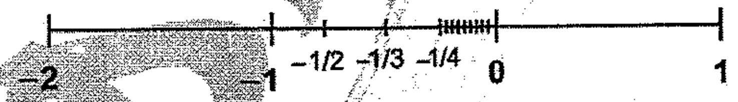


   Then $x \le 1 \ \forall x \in S$

   $\therefore$ upper bound of S is 1

2. **Lower bound:** Let S be any subset of $\mathbb{R}$ then a real number m is called **lower bound** of S if every element of S is greater than or equal to m i.e. $m \le x \quad \forall x \in S$.

   Example: $S = \left\{-\frac{1}{n}; n \in \mathbb{N}\right\}$

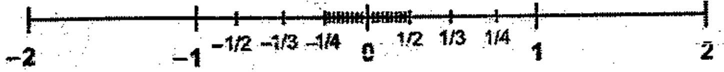


   Then $-1 \le x \ \forall x \in S$

   $\therefore$ lower bound of S is -1

3. **Bounded Set:** A set $S \subseteq \mathbb{R}$ is called bounded if $\exists$ two real numbers m and M such that $m \le x \le M \ \forall x \in S$

   Example: $S = \left\{\frac{1}{n}; n \in \mathbb{N}\right\} \cup \left\{-\frac{1}{n}; n \in \mathbb{N}\right\}$



   Then $-1 \le x \le 1 \ \forall x \in S$

   $\therefore S$ is bounded.

4. **Greatest Element:** If an upper bound of a set belongs to the set then it is called greatest element of the set.

   Examples:

   (a) $S=\left\{\frac{1}{n}:n\in\mathbb{N}\right\}$, 1 is the Greatest element of S.

   (b) $S=(0,2]$, 2 is the Greatest element of S.

5. **Smallest Element:** If lower bound of a set belongs to the set then it is called smallest element of the set.

   Examples:

   (a) $S=\left\{\frac{1}{n}:n\in\mathbb{N}\right\}\cup\{0\}$, 0 is the least element of S.

   (b) $S=[2,3]$, 2 is the least element of S.

6. **Supremum of a set:** Let $S\subseteq\mathbb{R}$, then smallest of all the upper bounds of a set S is called the **Supremum** of the set S or the **least upper bound** $(l.u.b.)$ of the set S.

   Examples:

   (a) $S=\left\{1,1+\frac{1}{2},1+\frac{1}{2}+\frac{1}{2^2},1+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots\right\}$

   $\sup S=2\,\&\,2\notin S$

   (b) $S=\left\{\frac{4n+3}{n};n\in\mathbb{N}\right\}$

   $\sup S=7\,\&\,7\in S$.

7. **Infimum of the set:** Let $S\subseteq\mathbb{R}$, Then greatest of all the lower bounds of a set S is called the **Infimum** of the set S or the **greatest lower bound** $(g.l.b)$ of the set S.

   Examples:

   (a) $S=\left\{\frac{4n+2}{n}:n\in\mathbb{N}\right\}$

   $\inf S=4\notin S$

   (b) $S=\left\{1,1+\frac{1}{2},1+\frac{1}{2}+\frac{1}{2^2}\ldots1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}\ldots\right\}$

   $\inf S=1\in S$

8. **Interval:** Let S be a subset of $\mathbb{R}$ then we say S is an interval if whenever $a,b\in S$ such that $a\neq b$. Then $\forall\ a<x<b\Rightarrow x\in S$.

   Note:

   (a) S is not an interval if there exist $a,b\in S,a\neq b$ and there exist t such that $a<t<b$ but $t\notin S$.

   (b) Empty Set and singleton Sets are intervals.

Put Your Own Notes

(c) If S is an interval and $\sup S=\beta$, $\inf S=\alpha$ then S is denoted by $(\alpha,\beta),[\alpha,\beta],[\alpha,\beta),(\alpha,\beta]$ (depending whether they are member of S or not.)

(d) Length of interval $=\sup S-\inf S$.

(e) An interval S is called an open interval if $\sup S$ and $\text{int} S$ both does not belong to S.

(f) An interval S is called closed interval if $\sup S$ and $\text{int} S$ both belong to S.

9. **Order Completeness Property of $\mathbb{R}$ (O.C.P.):** Every non empty bounded Subset of $\mathbb{R}$ has Supremum and Infimum in $\mathbb{R}$.

10. **Neighbourhood of a Point:** Let $S\subseteq\mathbb{R}, a\in\mathbb{R}$, we say S is a Neighbourhood of a, if there exist an open interval I such that $a\in I$ and $I\subseteq S$.

Examples:

(a) $\mathbb{R}$ is a neighbourhood of every real number.

(b) $\mathbb{N}$ is not a neighbourhood of any real number.

11. **Adherent point:** Let $a\in\mathbb{R}$ and $S\subseteq\mathbb{R}$, then a is called Adherent point of S, If for every $\delta>0, (a-\delta,a+\delta)\cap S\neq\phi$.

Examples:

(a) $S=\mathbb{N}$, Then Set of all adherent points of S is S itself.

(b) $S=(0,1]$, Then Set of all adherent points of S is $[0,1]$

12. **Isolated Point:** Let $a\in S$ and $S\subseteq\mathbb{R}$, then a is called Isolated point of S, if there exist $\delta>0$, such that $(a-\delta,a+\delta)\cap S=\{a\}$

Examples:

(a) $S=\mathbb{N}$, then every $a\in S$ is an Isolated point of S.

(b) $S=\mathbb{R}$, then no point of $\mathbb{R}$ is Isolated point of S.

13. **Limit Point:** Let $a\in\mathbb{R}$ and $S\subseteq\mathbb{R}$, then a is called Limit point of S, if for every $\delta>0, (a-\delta,a+\delta)\cap S-\{a\}\neq\phi$

Examples:

(a) $S=\left\{\frac{1}{n}:n\in\mathbb{N}\right\}$ then 0 is only limit point of S.

(b) $S=[0,1)$ then set of all limit points of S is $[0,1]$

14. **Interior Point:** Let $S\subseteq\mathbb{R}$ and $a\in\mathbb{R}$ we say a is an interior point of S if there exist $\delta>0$ such that $(a-\delta,a+\delta)\subset S$

Set of all interior points of S is denoted by S^o

Examples:

(a) $S=\mathbb{N}$, then no real number is an Interior point of S.

(b) $S=\mathbb{R}$ or $S=(0,1)$ then every member of S is an interior point of S.

15. **Exterior Point:** Let $a \in \mathbb{R}$ and $S \subseteq \mathbb{R}$, we say a is an exterior point of S if it is an interior point of S^c in $\mathbb{R}$ (where S^c means complement of S in $\mathbb{R}$). Set of all exterior point of S is denoted by $\text{Ext}\, S$.

Examples:

(a) $S = \mathbb{Q}$ then no real number which is Exterior point of $\mathbb{Q}$

(b) $S = [0,1]$ then set of all exterior Point of S is $(-\infty, 0) \cup (1, \infty)$

16. **Condensation Point:** Let $a \in \mathbb{R}$ and $S \subseteq \mathbb{R}$, then a is called Condensation point of S if for every $\delta > 0$, $(a-\delta, a+\delta)$ must contain uncountable members of S. Hence if S is not countable, then there exists a point in S such that x is condensation point of S

Example:

(a) $S = \mathbb{Q}^c$ then every Real number is a Condensation Point of S

(b) $S = \mathbb{Q}$ then no Real number is a Condensation Point of S.

17. **Frontier Point:** $a \in \mathbb{R}$ is said to be a frontier Point of $S \subseteq \mathbb{R}$ if it is neither an interior Point of S nor an exterior Point of S. Set of all frontier point of S is denoted by $\text{Fr}\, S$

Example:

(a) $S = \mathbb{R}$

$$S^o = \mathbb{R} \Rightarrow \left(S^o\right)^c = \phi$$

$$\text{Ext}\, S = \phi \Rightarrow (\text{Ext}\, S)^c = \mathbb{R}$$

$$\text{Now, } \text{Fr}(S) = \left(S^o\right)^c \cap (\text{Ext}\, S)^c = \phi \cap \mathbb{R} = \phi$$

(b) $S = \mathbb{N}$

$$S^o = \phi \Rightarrow (S^o)^c = \mathbb{R}$$

$$\text{Ext}\, S = \mathbb{R} - \mathbb{N} \Rightarrow (\text{Ext}\, S)^c = \mathbb{N}$$

$$\text{Fr}\, S = \mathbb{R} \cap \mathbb{N} = \mathbb{N}$$

18. **Boundary Point:** Frontier Point of S which are member of S are referred as boundary point. Set of all boundary point of S is denoted by ∂S

Example:

(a) $S = \mathbb{R}$

Then $\partial S = \text{Fr}\, S \cap S = \phi \cap S = \phi$

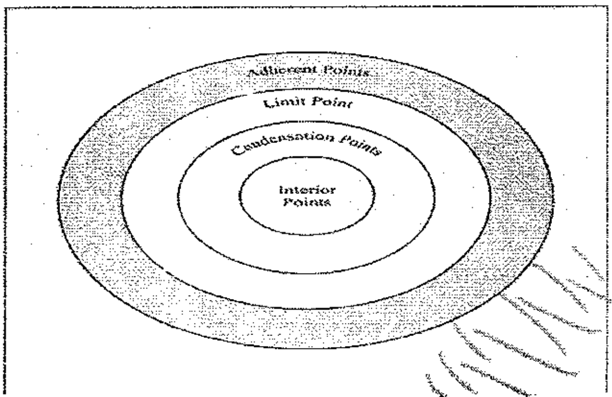
(b) $S = \mathbb{N}$

$$\partial S = \text{Fr}\, S \cap S = \mathbb{N} \cap \mathbb{N} = \mathbb{N}$$

19. **Open Set:** A set S is called open, if it is neighbourhood of each of its point.

20. **Closed Set:** A set S is called closed, if it contains all its limit point.

21. **Derived Set:** Collection of all limit point of a given set S is called derived set of S and it is denoted by S'
22. **Perfect Set:** If $S = S'$ then S is called perfect set.
23. **Dense Set:** A set S is said to be Dense in any set X iff $\overline{S} = X$.



2.2. Bolzano Weierstrass Theorem

Every infinite bounded subset of $\mathbb{R}$ has a limit point.

Note:

(a) This is only a sufficient condition for a set to have a limit point i.e. unbounded infinite sets may have limit point

Example: $S = \mathbb{R}$

(b) Condition of infiniteness cannot be dropped.

Example: $S = \{0,1\}$ does not have any limit point.

(c) Condition of boundedness cannot be relaxed.

Example: $S = \mathbb{Z}$ does not have any limit point.

2.3. Properties

1. A set may or may not posses upper and/or lower bounds.

   Example:

   (a) $S = (-\infty, 0)$ then upper bound of S is 0 and lower bound of S does not exist

   (b) $S = [0, \infty)$ then upper bound of S does not exist and Lower bound of S is 0

   (c) $S = [0,1]$ then upper bound of S is 1 and lower bound of S is 0

2. Bounds of a set may or may not belong to the set.
3. If a set has one upper bound / lower bound then it has an infinite number of upper bounds/lower bounds.
4. If S is a bounded non-empty subset of $\mathbb{R}$. Then $\inf S \le \sup S$

5. If $S \subseteq T \subseteq R$ where $S \neq \phi$ then
   (a) If T is a bounded above then $\sup S \leq \sup T$ (existence of sup is followed by O.C.P.)
   (b) If T is bounded below then $\inf S \geq \inf T$
6. For any two non-empty bounded subsets S, T of $\mathbb{R}$ such that $S \subseteq T$; $\inf T \leq \inf S \leq \sup S \leq \sup T$
7. For any two non-empty subsets S, T of $\mathbb{R}$ such that $s \leq t \ \forall s \in S, t \in T$ then S is bounded above and T is bounded below and $\sup(S) \leq \inf T$.
8. If S and T be arbitrary non-empty bounded subsets of $\mathbb{R}$ then
   (a) $\sup(S \cup T) = \max\{\sup S, \sup T\}$
   (b) $\inf(S \cup T) = \min\{\inf S, \inf T\}$
9. If S and T are two non-empty subset of $\mathbb{R}$ then $\sup(S \cap T) \neq \min\{\sup S, \sup T\}$

   Example:

   $S = [0,1] \cup \{4\}$ and $T = [0,2]$ then $\sup S = 4$ and $\sup T = 2$ and $\min\{\sup S, \sup T\} = 2$

   $S \cap T = [0,1]$ and $\sup(S \cap T) = 1$. Thus $\sup(S \cap T) \neq \min\{\sup S, \sup T\}$
10. Let S be any non-empty subset of $\mathbb{R}$ having Supremum and let $T = \{\lambda x : x \in S\}$ then T is bounded above $\sup T = \lambda \sup S \ \forall \lambda \geq 0$
11. Let S be a non-empty set of $\mathbb{R}$ which has Supremum and let $T = \{\lambda x : x \in S\}$ then T is bounded below and $\inf T = \lambda \sup S \ \forall \lambda \leq 0$
12. $\sup\{x \in \mathbb{Q} : x < a\} = a$ for each $a \in \mathbb{R}$.
13. Finite union and arbitrary intersection of bounded subsets of $\mathbb{R}$ is bounded.
14. A subset of bounded set is bounded.
15. If S and T are two bounded sets Then
   (a) $\sup\{x + y : x \in S, y \in T\} = \sup S + \sup T$
   (b) $\inf\{x + y : x \in S, y \in T\} = \inf S + \inf T$
   (c) $\sup\{x - y : x \in S, y \in T\} = \sup S - \inf T$

(d) $\inf\{x - y : x \in S, y \in T\} = \inf S - \sup T$

(e) $\sup\{xy : x \in S, y \in T\} = \sup S \sup T$ (where S and T are bounded subset of positive real numbers)

(f) $\inf\{xy : x \in S, y \in T\} = \inf S \inf T$ (where S and T are bounded subset of positive real numbers)

16. Let\ S be a non–empty subset of $\mathbb{R}$ having supremum and $T = \{x \in \mathbb{R} : x - a \in S\}$, $C = \left\{x \in \mathbb{R} : \frac{x + a}{2} \in S\right\} \forall a \in \mathbb{R}$ then $\sup T = a + \sup S$ and $\sup C = 2\sup S - a$

Example:

Let $S = \{1,2,3,4,5\}$ then $\sup S = 5$ and

$T = \{x \in \mathbb{R} : x - 5 \in S\} = \{6,7,8,9,10\}$

Clearly $\sup T = 10 = 5 + \sup S$

Define

$C = \left\{x \in \mathbb{R} : \frac{x+5}{2} \in S\right\} = \{5,3,1,-1,-3\}$

Clearly $\sup C = 5 = 2\sup S - 5$

17. Every bounded subset of $\mathbb{R}$ is contained in an interval of finite length.
18. Countably infinite set is not a neighbourhood of any of its points.
19. Non-empty finite set is not a neighbourhood of any of its points.
20. Empty set ϕ is neighbourhood of each of its points.
21. Every neighbourhood of a point is uncountable.
22. Uncountability is necessary condition for a set to be neighbourhood of a point but not sufficient.

Example: $S = \mathbb{Q}^c$ then S is an uncountable set but not neighbourhood of any of its points.

23. Superset of a neighbourhood is also neighbourhood of same point.
24. Subset of a neighbourhood may or may not be a neighbourhood of same point.

Example: $S = [-1,1]$ is neighbourhood of 0 but it's subset $T = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is not.

25. Arbitrary union of family of neighbourhood is also neighbourhood.
26. If S and T are two neighbourhoods of point $p \in \mathbb{R}$, then $S \cap T$ is also a neighbourhood of p
27. Finite intersection of neighbourhoods of a point is also neighbourhood of that point.
28. Arbitrary intersection of neighbourhoods need not be a neighbourhood of a Point.

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Example: $S_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$

Then each S_n is a neighbourhood of 0.

But $\bigcap_{n=1}^{\infty} S_n = \{0\}$ is not a neighbourhood of 0.

29. There exist a countable infinite set $S \subseteq \mathbb{R}$, such that $\sup S \in S^c$ and inf $S \in S^c$

Example: $S = \left(\sqrt{2}, \sqrt{3}\right) \cap \mathbb{Q}$ then $\sup S = \sqrt{3} \in S^c$ and $\inf S = \sqrt{2} \in S^c$.

30. There exist a uncountable set $S \subseteq \mathbb{R}$, such that $\sup S \in S^c$ and $\inf S \in S^c$.

Example: $S = (0,1)$ then $\sup S = 1 \in S^c$ and $\inf S = 0 \in S^c$

31. There exist countably infinite set $S \subseteq \mathbb{R}$, such that $\sup S \in S$ and inf $S \in S^c$

Example: $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ then $\sup S = 1 \in S$ and $\inf S = 0 \in S^c$

32. There exist uncountable set $S \subseteq \mathbb{R}$, such that $\sup S \in S$ and $\inf S \in S^c$

Example: $S = (0,1]$ then $\sup S = 1 \in S$ and $\inf S = 0 \in S^c$

33. Intersection of the family of all neighbourhoods of a point x is singleton set $\{x\}$
34. A finite set has no limit point
35. Limit point of a set may or may not belong to the set

Example: $S = (0,1]$ then 0 is a limit point of S. But $0 \notin S$ also 1 is a limit point of S and $1 \in S$.

36. Infiniteness is the necessary condition for a set to have a limit point but not sufficient.

i.e. Infinite set may not have a limit point

Example: $S = \mathbb{N}$ (Set of all Natural number), does not have a limit point.

37. Uncountable set must have limit point.
38. Set of Isolated points of a set is always countable.
39. Every limit point of a set is an adherent point of that set but converse may not be true.

Example: $S = \mathbb{N}$ (Set of all Natural number)

Then every member of S is an Adherent Point of S. But no member of S is a limit point of S

40. Limit point of a set S cannot be interior point of S^c

Reason: Let $p \in \mathbb{R}$ be a limit point of the set $S \subseteq \mathbb{R}$ and p is also an interior point of S^c

Then $\exists\ \varepsilon > 0$ such that $(p-\varepsilon, p+\varepsilon) \subset S^c$ then $(p-\varepsilon, p+\varepsilon) \cap S = \phi$ thus p is not a limit point of S which contradicts the hypothesis.

41. Every interior point of a set is limit point of that set. Conserve may not be true.

Example: $S = (0,1)$ then 0 is a limit point of the sets but not an interior point.

42. Every interior point of set is adherent point of a set. Converse may not be true

Example: $S = (0,1]$ then 0 is an adherent point of S but not an interior point of S.

43. Interior point of S^c cannot be limit point of S.

44. Supremum of a set S (if exist) is either greatest member if S or a limit point of S

45. Infimum of a set S (if exist) is either least member of S or a limit point of S

46. For every set S, S° is an open set.

47. S° is the largest open set contained in S

48. S and T are two set. Then

(i) $S \subseteq T \Rightarrow S^\circ \subseteq T^\circ$

(ii) $(S \cap T)^\circ = S^\circ \cap T^\circ$

(iii) $(S \cup T)^\circ \supset S^\circ \cup T^\circ$

49. For any subset S of $\mathbb{R}$, $S^\circ \subset S$.

50. S is open iff $S = S^\circ$, for any subset S of $\mathbb{R}$.

51. For any set S; $(S^\circ)^\circ = S^\circ$.

52. S° is equal to the union of all open subsets of S

53. A set is closed iff its complement is an open set.

54. The intersection of arbitrary family of closed sets is closed

55. Finite union of closed set is closed.

56. Arbitrary union of closed sets may not be closed.

Example: $S_n = \left[\frac{1}{n}, 1-\frac{1}{n}\right] \forall n \geq 2$ then each S_n is closed set but $\bigcup_{n=1}^{\infty} S_n = (0,1)$ is an open set.

57. Every finite set is a closed set.

58. Derived set is always closed set

59. Derived set of a bounded set is bounded set.

60. If derived set (say) S' is finite then S is countable set.

61. *Let* S *and* T *are two subset of real numbers. Then*

    (i) $S \subset T \Rightarrow S' \subset T'$

    (ii) $(S \cup T)' = S' \cup T'$

    (iii) $(S \cap T)' \subset S' \cap T'$ (where S', T' denote the derived set of S and T respectively)

62. Non-empty countable set cannot be open

63. If a set S is open then $S \subseteq S'$ converse need not be true

    Example: $S = (0,1]$ then $S \subseteq S' = [0,1]$ but S is not an open set.

64. If S is an open set and T is closed set. Then $S - T$ is open and $T - S$ is closed set.

65. Arbitrary intersection of open sets may not be open.

    Example: $S_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ then each S_n is an open set. But $\bigcap_{n=1}^{\infty} S_n = \{0\}$ which is not an open set.

66. Every non-empty open set is union of open intervals.

67. Closure of a set S is denote by $\overline{S}$ and $\overline{S} = S \cup S'$ i.e. closure is union of S and set of limit points of S

Some Examples:

Set	Supremum	Infimum	Set of adherent points
$\mathbb{N}$	does not exist	1	$\mathbb{N}$
$\mathbb{Z}$	does not exist	does not exist	$\mathbb{Z}$
$\mathbb{Q}$	does not exist	does not exist	$\mathbb{R}$
$\mathbb{Q}^C$	does not exist	does not exist	$\mathbb{R}$
$\mathbb{R}$	does not exist	does not exist	$\mathbb{R}$
$\left\{\frac{1}{n}; n \in \mathbb{N}\right\}$	1	0	$\left\{\frac{1}{n}; n \in \mathbb{N}\right\} \cup \{0\}$

$[a,b)\,;a\neq b$	b	a	$[a,b]$
$\left\{x+\frac{1}{n};n\in\mathbb{N}\right\}$	$x+1$	x	$\left\{x+\frac{1}{n};n\in\mathbb{N}\right\}\cup\{x\}$
$\{\cos n; n\in\mathbb{N}\}$	1	-1	$[-1,1]$

Set	Set of limit points	Frontier points	Boundary points	Nbd. of
$\mathbb{N}$	ϕ	$\mathbb{N}$	$\mathbb{N}$	None of its Points
$\mathbb{Z}$	ϕ	$\mathbb{Z}$	$\mathbb{Z}$	None of its Points
$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{Q}$	None of its Points
$\mathbb{Q}^C$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{Q}^C$	None of its Points
$\mathbb{R}$	$\mathbb{R}$	ϕ	ϕ	Each of its Points
$\left\{\frac{1}{n};n\in\mathbb{N}\right\}$	$\{0\}$	$\left\{\frac{1}{n};n\in\mathbb{N}\right\}$	$\left\{\frac{1}{n};n\in\mathbb{N}\right\}$	None of its Points
$[a,b)\,;$ $a\neq b$	$[a,b]$	$\{a,b\}$	$\{a\}$	(a,b)
$\left\{x+\frac{1}{n};n\in\mathbb{N}\right\}$	$\{x\}$	$\left\{x+\frac{1}{n};n\in\mathbb{N}\right\}$	$\left\{x+\frac{1}{n};n\in\mathbb{N}\right\}$	None of its Points
$\{\cos n; n\in$	$[-1,1]$	$\{\cos n; n\in\mathbb{N}\}$	$\{\cos n; n\in\mathbb{N}\}$	None of its Points

****Students are advised to extend this table by themselves****

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CHAPTER 3

SEQUENCES OF REAL NUMBERS

3.1. Definition

A map from $\mathbb{N}$ to $\mathbb{R}$ defines a sequence of real numbers when the images are arranged in the natural order of natural number i.e., starting with images of n is followed by that of $(n+c)$ i.e., $f:\mathbb{N}\to\mathbb{R}$ is a sequence of real numbers and $<f(n)> = <f(1), f(2), \ldots>$ we write $<f(n)> = <a_n><b_n>, <u_n>, <v_n>$ etc. $\forall n \in \mathbb{N}$, $<a_n> = <a_1, a_2, a_3, \ldots>$.

When $f:\mathbb{N}\to\mathbb{C}$, then this function defines a sequence of complex numbers.

Note: A map $f:S_{(n)}\to\mathbb{R}$ (1). Define a finite sequence of length n called as string ($S_{(n)}$ denote the set of first n natural numbers). Generally sequence is always infinite (i.e., non-terminating) but if defined on finite set then it is given by (1).

3.1.1. Range set of a Sequence

The range set of a sequence is the set consisting of all distinct elements of a sequence and without regard to the position term. Thus the range may be finite or infinite set i.e., the range set of sequence $<u_n>$ is given by $\{u_n : n\in\mathbb{N}\}$ or simply by $R(u)$.

Some Important Example of Sequence

1. $<a_n> = <n>$
2. $<a_n> = <-n>$
3. $<a_n> = (-1)^n n$
4. $<a_n> = (-1)^n$
5. $<a_n> = \frac{1}{n}$
6. $a_n = 1 - \frac{(-1)^n}{n}$
7. $a_n = \begin{cases} 2 & n=1 \text{ or prime} \\ n & \text{else} \end{cases}$
8. $a_n = \begin{cases} 2 & n=1 \text{ or prime} \\ p & p/n \text{ \& p is the least prime} \end{cases}$

9. $a_{n+2} = a_n + a_{n+1}, \quad a_1 = 1, \quad a_2 = 1$

10. $a_{n+2} = \frac{1}{2}\left(a_n + a_{n+1}\right)$ where a_1 & a_2 given

11. $a_{n+1} = \sqrt{2 + a_n}$ where $a_1 = 0$

12. $< a_n > = 2012 \quad \forall n \in \mathbb{N}$

13. $< a_n > = (-1)^n + \sin\left(\frac{n\pi}{6}\right)$

14. $< a_n > = \begin{cases} -1 - \frac{1}{n} & n \text{ is even} \\ 1 + \frac{1}{n} & n \text{ is odd} \end{cases}$

15. $< a_n > = \cos n$

16. $< a_n > = \sin\left(\frac{1}{n}\right)$

17. $< a_n > =$ Last digit of 3^n

18. $< a_n > = \{-2, -3, -1, 0, 1, 2, 1, 0, 1, 0, \ldots\}$

19. $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n$

20. $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

3.1.2. Bounded Sequence

A sequence $< u_n >$ is defined as bounded if its range set is bounded. Hence $< u_n >$ is bounded if there exist real numbers k' and k such that $k' \le u_n \le k \ \forall \ n \in \mathbb{N}$.

OR

If there exist $K \ge 0$ such that $|u_n| \le K \ \forall \ n \in \mathbb{N}$.

Note:

(a) Supremum and infimum of the range set is the supremum and infimum of the sequence.

(b) If the range of the sequence $< a_n >$ is finite then there exist $\alpha \in \mathbb{R}$ such that $a_n = \alpha$ for infinitely many values of n. however converse need not be true.

Example: $a_n = \begin{cases} 4 & n = 1 \text{ or prime} \\ n & \text{else} \end{cases}$

Then $a_n = 4$ for infinitelymany values of n but range of the sequence $< a_n >$ is not finite

3.1.3. Monotonic Sequence

Let $< a_n >$ be a sequence of real numbers $< a_n >$ is monotonic if either $a_n \leq a_{n+1}\ \forall n \in \mathbb{N}$ (Then the sequence is called monotonically increasing or non-decreasing) or $a_n \geq a_{n+1}\ \forall n \in \mathbb{N}$ (Then the sequence is called monotonically decreasing or non-increasing).

If $a_n < a_{n+1}\ \forall n \in \mathbb{N}$, then $< a_n >$ is called stricly increasing or increasing sequence.

If $a_n > a_{n+1}\ \forall n \in \mathbb{N}$, then $< a_n >$ is called stricly decreasing or decreasing sequence.

Examples:

(a) $< a_n > = < n >$ is increasing sequence.

(b) $< a_n > = < -n >$ is decreasing sequence.

(c) $a_{n+1} = \sqrt{2 + a_n}\ \ \forall n \in \mathbb{N}$ is monotonically increasing sequence.

3.1.4. Eventual Nature of a Sequence

1. A sequence $< a_n >$ is said to be eventually monotonically increasing if $\exists\ m \in \mathbb{N}$ such that $a_{n+1} \geq a_n\ \ \forall n \geq m$.

   Example: $< 0, 2, -5, 3, 7, 8, 9, 10, 11, 12, \ldots >$

2. A sequence $< a_n >$ is said to be eventually monotonically decreasing if $\exists\ m \in \mathbb{N}$ such that $a_{n+1} \leq a_n\ \forall n \geq m$.

   Example: $\left\langle 0, 2, 0, 2, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\rangle$

3. A sequence $\langle a_n \rangle$ is eventually constant sequence if $\exists\ m \in \mathbb{N}$ such that $a_n = c\ \ \forall\ n \geq m$ where $c \in \mathbb{R}$.

   Example: $< 0, 2, 0, 2, 0, 3, 8, 5, 7, 7, 7, 7, 7, 7, \ldots >$

3.2. Limit Point of a Sequence

A real number p is said to be a **limit point** or a **cluster point** of a sequence if every neighbourhood of p contains an infinite number of elements of the given sequence.

In other words, a real number p is a **limit point** of a sequence $\langle a_n \rangle$ if for any $\varepsilon > 0$, $a_n \in (p - \varepsilon, p + \varepsilon)$ for infinitely many values of n.

Remark: A real number p is **not a limit point** of a sequence $\langle a_n \rangle$, if there exists at least one neighbourhood of p which contains only **finite** number of elements of $\langle a_n \rangle$.

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Note:

(a) Limit point of a sequence need not be a limit point of the range set.

Example: $< a_n > = (-1)^n$

Then $\{1,-1\}$ is a set of limit points of $< a_n >$ but neither 1 nor-1 limit point of its range set.

(b) Limit point of the sequence can be obtained in three ways and they are as follows :

(a) Representation on real line.

(b) If the range set has a limit point, then sequence has limit point.

(c) If the range set is finite.

$\Rightarrow a_n = \alpha$ for infinitely many values of n where $\alpha \in \mathbb{R}$.

$\Rightarrow \alpha$ is a limit point of the sequece.

3.2.1. Existence of A Limit Point

Bolzano Weierstrass Theorem: Every bounded sequence has a limit point.

Reason: Since the sequence ($< a_n > say$) is bounded thus Range set of $< a_n >$ is bounded.

Case I: If range set is infinite then by Bolzano Weierstrass Theorem for set, range set must have limit point thus sequence $< a_n >$ has a limit point.

Case II: If range set is finite then $\exists\ \alpha \in \mathbb{R}$ such that $a_n = \alpha$ for infinitely many values of n .Evidently, α is a limit point of the sequence $< a_n >$.

Hence, every bounded sequence has a limit point.

Application:

1. $< a_n > = (-1)^n$ has limit points 1,-1

2. $< b_n > = < \frac{1}{n} >$ has limit point 0

3. $< c_n > = \left(1+\frac{1}{n}\right)^n$ has limit point e

3.3. Subsequence

Subsequence of a sequence is defined with the help of sequence of natural number. Consider a map $f : \mathbb{N} \to \mathbb{N}$ defined by $f(k) = n_k$ and let $< n_k >$ be a strictly increasing sequence of natural numbers. Then for any sequence $< a_n >$, the sequence $< a_{nk} >$ is defined as a subsequence of $< a_n >$.

Example: Let $< a_n >$ be any sequence of real number. Then $< a_{2k} >, < a_{2k-1} >, < a_{2^k} >, < a_{3k} >$ all are subsequences of $< a_n >$ where $< a_{2k} > = < b_n > = b_1, b_2, b_3, \ldots = a_2, a_4, a_6, a_8, \ldots$

Similarly $< a_{2^k} > = < c_n > = c_1, c_2, c_3, c_4, \ldots = a_2, a_4, a_8, a_{16}, \ldots$

Thus $< b_n >$ is a sub sequence of $< a_n >$ and $< c_n >$ is a subsequence of $< b_n >$.

3.3.1. Complementary Subsequences

Let $< a_{n_k} >$ and $< a_{n_{k'}} >$ are subsequence of $< a_n >$ then define $S_1 = \{n_k \mid n_k \in \mathbb{N}\}$ and $S_2 = \{n_{k'} \mid n_{k'} \in \mathbb{N}\}$, then $< a_{n_k} >$ and $< a_{n_{k'}} >$ are complementary subsequences if

1. $S_1 \cup S_2 = \mathbb{N}$
2. $S_1 \cap S_2 = \phi$

Example: If $< a_n >$ is a sequence of real numbers. Then $< a_{2n} >$ and $< a_{2n-1} >$ are complementary subsequence of $< a_n >$.

Remark: Unbounded sequence may have limit point.

3.4. Limit of a Sequence

Let $< a_n >$ be a sequence of real number and l be a real number, $(l \in \mathbb{R})$ then we say l is the limit of the sequence $< a_n >$ if for any $\in > 0$ $\exists m \in \mathbb{N}$ such that $|a_n - l| < \in \ \forall n \geq m$.

Symbolically, we write it as $\lim\limits_{n \to \infty} a_n = l$ or $a_n \to l$ as $n \to \infty$.

Algorithm: To claim l is limit of a sequence $\langle a_n \rangle$

1. Simplify $|a_n - l|$ to get $|a_n - l| \leq \phi(n)$, where $\phi(n)$ is function of n.
2. Solve the inequality $\phi(n) < \in$ to get $n > \psi(\in)$.
3. If $\psi(\in)$ is defined $\forall \in > 0$ then set $m = [\psi(\in)] + 1$ (where $[\psi(\in)]$ denote the ceiling function)

   $\Rightarrow |a_{n-l}| < \in \ \forall n \geq m$.

Result Based on the Definition:

Let $\lim\limits_{n \to \infty} a_n = a$ and $\lim\limits_{n \to \infty} b_n = b$ $(a, b \in \mathbb{R})$, then

1. $\lim\limits_{n \to \infty} (a_n + b_n) = \lim\limits_{n \to \infty} a_n + \lim\limits_{n \to \infty} b_n = a + b$
2. $\lim\limits_{n \to \infty} (a_n - b_n) = \lim\limits_{n \to \infty} a_n - \lim\limits_{n \to \infty} b_n = a - b$
3. $\lim\limits_{n \to \infty} (a_n . b_n) = \lim\limits_{n \to \infty} (a_n) \lim\limits_{n \to \infty} (b_n) = a.b$
4. $\lim\limits_{n \to \infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{\left(\lim\limits_{n \to \infty} a_n\right)}{\left(\lim\limits_{n \to \infty} b_n\right)} = \dfrac{a}{b}$, provided $b_n \neq 0 \ \forall n \in \mathbb{N}$ and $b \neq 0$.
5. $\lim\limits_{n \to \infty} a_n = a \Leftrightarrow \lim\limits_{n \to \infty} (ka_n) = k \lim\limits_{n \to \infty} (a_n) = k.a \qquad \forall k \in \mathbb{R} - \{0\}$

Remarks:

(a) In view the results 1–5 we can say that the operation of limit is compatible with addition subtraction multiplication divison and scaler multiplication.

(b) Converse of the above results 1–4 is not true.

Examples:

(a) Take $< a_n > = < n >$ and $< b_n > = < -n >$, then $< a_n >, < b_n >$ are not convergent

(i) $< a_n + b_n > = < 0 >$, which is convergent.

(ii) $\dfrac{< a_n >}{< b_n >} = < -1 >$, which is convergent.

(b) Take $< a_n > = (-1)^n, < b_n > = (-1)^{n+1}$, then $< a_n >, < b_n >$ both are not convergent

(i) $< a_n . b_n > = < (-1)^{2n+1} > = < -1 >$ Which is convergent.

(ii) $< a_n - b_n > = < 0 >$ Which is convergent.

(c) Every convergent sequence is bounded; however converse need not be true.

Reason: Let $\lim\limits_{n \to \infty} a_n = a \qquad (a \in \mathbb{R})$

Then for given $\in > 0$, $\exists\ k \in \mathbb{N}$ such that

$|a_n - a| < \in \ \forall n \geq k \Rightarrow a - \in < a_n < a + \in \ \forall n \geq k$

Take $M = \max\{a_1, a_2, ..., a_{k-1}, a + \in\}$ and $m = \min\{a_1, a_2, ..., a_{k-1}, a - \in\}$

Then $m \leq a_n \leq M \ \ \forall n \in \mathbb{N}$. This implies $< a_n >$ is bounded.

Note: Converse need not be true.

Example: Take $< a_n > = (-1)^n$, then $< a_n >$ is bounded but not convergent.

(c) Limit of a convergent sequence is unique.

Reason: Let $< a_n >$ be a sequence of real number such that $\lim\limits_{n \to \infty} a_n = a$

Let if possible, $< a_n >$ also converges to $a' \ (a' \in \mathbb{R})$

Then for given $\in > 0$, $\exists\ m_1, m_2 \in \mathbb{N}$ such that

$|a_n - a| < \in /2 \quad \forall n \geq m_1$

$|a_n - a'| < \in /2 \quad \forall n \geq m_2$

Take $M = Max\{m_1, m_2\}$

Now, $|a - a'| = |-a_n + a + a_n - a'|$

$\Rightarrow |a - a'| \leq |a_n - a| + |a_n - a'| \ \forall\ n \geq m$

$< \in /2 + \in /2 = \in \ \forall n \geq m$

$\Rightarrow |a - a'| < \in$

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Since $\in$ was arbitrary small. So $\in \to 0$, $a = a'$

Hence, limit of sequence $\langle a_n \rangle$ is unique.

Note:

(i) l is a limit $\Leftrightarrow$ every Neighbourhood of l leaves out only finite members of the sequence.

(ii) Limit of a sequence is limit point but converse need not be true.

(d) If $\lim\limits_{n\to\infty} u_n = l \Rightarrow \lim\limits_{n\to\infty} |u_n| = |l|$ Converse need not be true.

Example: Take $\langle u_n \rangle = (-1)^n$ then $\langle |u_n| \rangle = \langle 1 \rangle$, is convergent but $< u_n >$ is not convergent.

(e) $\lim\limits_{n\to\infty} |u_n - l| = 0 \Leftrightarrow \lim\limits_{n\to\infty} u_n = l$.

(f) If $|u_n| \le |v_n| \ \forall n \ge m$ and $\lim\limits_{n\to\infty} |v_n| = 0$ thus $\lim\limits_{n\to\infty} |u_n| = 0$ (where $|u_n|$ denotes the absolute value of u_n).

(g) If $\lim\limits_{n\to\infty} a_n = a$ and $a_n \ge b \quad \forall n \ge m, m \in \mathbb{N}$. Then $a \ge b \ (a, b \in \mathbb{R})$.

(h) A bounded sequence $< a_n >$ is convergent $\Leftrightarrow$ it has unique limit point.

Note:

(i) The above statement of the result provides a second definition for a sequence to be convergent i.e., a sequence is convergent if it is bounded and has a unique limit point.

(ii) Condition of the above result cannot be relaxed i.e., if a bounded sequence have more than one limit point then it cannot be convergent.

Example: A sequence $< a_n > = (-1)^n$, is bounded having two limit points 1 and -1 thus it is not convergent sequence.

Application:

1. If a sequence $\langle a_n \rangle$ have more than one limit point then $\langle a_n \rangle$ cannot be convergent.

   Example:

   (a) $< a_n > = (-1)^n$ is not convergent as 1 and -1 are two limit points of $\langle a_n \rangle$.

   (b) $< b_n > = < (-1)^n + \frac{1}{n} >$ is not convergent as 1 and -1 are two limit points of $\langle b_n \rangle$.

2. An unbounded sequence cannot converge to a real number.

   Example:

   (a) $< a_n > = < n >$ is not convergent.

   (b) $< b_n > = < (-1)^n + n >$ is not convergent.

3. If a sequence has limit point, then it may not be convergent.

   Example:

   Consider a sequence $< a_n > = < 1, 0, 2, 0, 3, 0, 4, 0, ... >$ then 0 is a limit point of $< a_n >$ but $< a_n >$ is not convergent as it is unbounded.

(i) A monotonic sequence $< a_n >$ is convergent $\Leftrightarrow$ it is bounded.

Remark:

(i) Convergent sequence may not be monotonic.

Example:

$$a_n = 1 - \frac{(-1)^n}{n}$$

(ii) A bounded and monotonically increasing sequence converges to its supremum.

(iii) A bounded and monotonically decreasing sequence converges to its infimum.

(iv) An eventually monotonic sequence is converges $\Rightarrow$ it is bounded

(v) If $< a_n >, < b_n >$ be two sequences of real numbers such that $a_n \le a_{n+1} \le b_{n+1} \le b_n \ \forall n \ge m$ and $\lim\limits_{n \to \infty} (a_n - b_n) = 0$. Then $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} b_n$.

(vi) Monotonic sequence can have at most one limit point.

Applications:

(a) The sequence $< a_n > = < \frac{1}{n} >$ converges to 0 as it is monotonically decreasing sequence and 0 is its infimum.

(b) The sequence $< a_n > = < n >$ is not convergent as it is monotonically increasing sequence which is not bounded above.

(c) If $a_1 = \sqrt{3}$ and $a_{n+1} = \sqrt{3 + \sqrt{a_n}}$ $\forall n \in \mathbb{N}$, then $< a_n >$ converges.

(d) The sequence $\langle S_n \rangle$ defined by the recursion formula : $S_{n+1} = \sqrt{7 + S_n}, S_1 = \sqrt{7}$ converges to the positive root of $x^2 - x - 7 = 0$

Solution: We have $S_1 = \sqrt{7}$ and $S_2 = \sqrt{7 + \sqrt{7}}$.

Then $S_2 > S_1$. Let us suppose that $S_{k+1} > S_k$.

Then $7 + S_{k+1} > 7 + S_k \Rightarrow \sqrt{7 + S_{k+1}} > \sqrt{7 + S_k} \Rightarrow S_{k+2} > S_{k+1}$.

Thus, by the principle of mathematical induction, $S_{n+1} > S_n \ \forall n \in \mathbb{N}$, and so $\langle S_n \rangle$ is monotonically increasing.

Again $S_1 = \sqrt{7} < 7$. Let us suppose that

$S_k < 7 \ \Rightarrow \sqrt{7 + S_k} < \sqrt{7 + 7} < 7 \ \Rightarrow S_{k+1} < 7$.

Thus, by the principle of mathematical induction, $S_n < 7 \forall n$, and so $\langle S_n \rangle$ is bounded above.

Since $\langle S_n \rangle$ is monotonically increasing and bounded above, so $\langle S_n \rangle$ is convergent.

Let $\lim\limits_{n\to\infty} S_n = l$, so that $\lim\limits_{n\to\infty} S_{n+1} = l$. ...(1)

Clearly $l > 0$.

Now $S_{n+1} = \sqrt{7+S_n} \Rightarrow S_{n+1}^2 = 7 + S_n$.

Taking the limit as $n \to \infty$ and using (1), we get $l^2 = 7 + l \Rightarrow l^2 - l - 7 = 0$.

Hence l is a positive root of $x^2 - x - 7 = 0$.

(e) Let $\langle a_n \rangle$ be a sequence defined as $a_1 > 0$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, $n \geq 1$.

Then $\langle a_n \rangle$ is a bounded monotonic sequence.

Solution: From $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ we obtain $a_n^2 - 2a_n a_{n+1} + 2 = 0$, which is quadratic in a_n. Since this equation has real roots, therefore $a_{n+1}^2 - 2 \geq 0 \Rightarrow a_{n+1} \geq \sqrt{2}\ \forall n \in \mathbb{N}$.

Thus the sequence $\langle a_n \rangle$ is bounded below by $\sqrt{2}$.

Now

$$a_n - a_{n+1} = a_n - \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \frac{1}{2}\left(a_n - \frac{2}{a_n}\right) = \frac{1}{2}\left(\frac{a_n^2 - 2}{a_n}\right) \geq 0\ \forall n$$

$\therefore\ a_n \geq a_{n+1}$ or $a_{n+1} \leq a_n\ \forall n$

$\Rightarrow \langle a_n \rangle$ is a monotonically decreasing sequence.

Since $\langle a_n \rangle$ is a monotonically decreasing and bounded below sequence, therefore, $\langle a_n \rangle$ is convergent.

Let $\lim\limits_{n\to\infty} a_n = l$, so that $\lim\limits_{n\to\infty} a_{n+1} = l$. Using in $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, we get

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{2}\left(\lim_{n\to\infty} a_n + \lim_{n\to\infty} \frac{2}{a_n}\right)$$

$$\Rightarrow l = \frac{1}{2}\left(l + \frac{2}{l}\right) \Rightarrow l^2 = 2 \Rightarrow l = \sqrt{2}. \qquad (\because l > 0)$$

Hence $\lim\limits_{n\to\infty} a_n = \sqrt{2}$.

(f) Let $\langle s_n \rangle$ be a sequence defined by $s_1 = 1$, $s_{n+1} = \left(\frac{3 + s_n^2}{2}\right)^{1/2}$ $\forall n \in \mathbb{N}$, then $\langle s_n \rangle$ is a bounded and monotonically increasing sequence and converges to $\sqrt{3}$.

(g) Let x_1, y_1 , are two positive unequal numbers and $x_n=\frac{1}{2}\left(x_{n-1}+y_{n-1}\right)$ and $y_n=\sqrt{x_{n-1}y_{n-1}}$ $\forall\ n\ge 2$ Then the sequence $\langle x_n\rangle$ and $\langle y_n\rangle$ are monotonic and they coverge to the same limit.

(j) Every sequence $\langle a_n\rangle$ has a monotonic subsequence $\left\langle a_{n_k}\right\rangle$.

(k) Let $\langle a_n\rangle$ be a sequence converging to l. Then every sub sequence of $<a_n>$ also converges to l.

Reason: Let $<a_n>$ be a sequence of real number which converge to l.

Let $<a_{n_k}>$ be a subsequence of $<a_n>$

Since $<a_n> \to l$

So for given $\in > 0$, $\exists\, m \in \mathbb{N}$ such that

$|a_n - l| < \in \qquad \forall n \ge m$

$\Rightarrow |a_{n_k} - l| \in \qquad \forall n_k \ge m$

$\Rightarrow <a_{n_k}> \to l.$

Applications:

(a) Let $<a_n> = <\frac{1}{n}>$ be a sequence then $\lim\limits_{n\to\infty} a_n = 0$. Now $\langle b_n\rangle = \left\langle \frac{1}{2n}\right\rangle, \langle c_n\rangle = \left\langle \frac{1}{3n}\right\rangle$ are subsequences of $\langle a_n\rangle$ so $\lim\limits_{n\to\infty} b_n = 0, \lim\limits_{n\to\infty} c_n = 0$. (By above result)

(b) Let $\langle a_n\rangle = \left\langle \sin\left(\frac{1}{n}\right)\right\rangle$ be a sequence then $\lim\limits_{n\to\infty} a_n = 0$. Now $\langle b_n\rangle = \left\langle \sin\left(\frac{1}{2n+1}\right)\right\rangle$ is a subsequence

$\langle a_n\rangle$ So by above result, $\lim\limits_{n\to\infty} b_n = 0$.

(l) A real number l is a limit point of a sequence $\langle a_n\rangle \Leftrightarrow$ there exist a subsequence $\left\langle a_{n_k}\right\rangle$ of $\langle a_n\rangle$ converges to l.

Remark:

(a) Every bounded sequence has a convergent subsequence.

(b) If each of the two subsequence $\langle a_{2n-1}\rangle$ and $\langle a_{2n}\rangle$ of a sequence $\langle a_n\rangle$ converges to l. Then $\langle a_n\rangle$ also converges to l. ($\because$ $\langle a_{2n-1}\rangle$ and $\langle a_{2n}\rangle$ are complementary subsequences of $\langle a_n\rangle$)

(c) If $\langle a_{2n-1}\rangle$ and $\langle a_{2n}\rangle$ converges to different limit then $\langle a_n\rangle$ cannot converge.

Example:

(a) Let $\langle a_n \rangle = \left\langle (-1)^n \right\rangle$.

Since $\langle a_n \rangle$ has two limit point -1 and 1.

$\therefore$ There must exist two different subsequences of $\langle a_n \rangle$

Say $\langle b_n \rangle$ and $\langle c_n \rangle$ such that $\lim\limits_{n\to\infty} b_n = 1$ and $\lim\limits_{n\to\infty} c_n = -1$.

Hence $\langle a_n \rangle$ is not convergent.

(b) Let $\langle a_n \rangle = \left\langle (-1)^n + \frac{1}{n} \right\rangle$

Since $\langle b_n \rangle = \left\langle (-1)^{2n} + \frac{1}{2n} \right\rangle$ is a subsequence of $\langle a_n \rangle$ and $\langle b_n \rangle \to 1$.

$\therefore$ 1 is a limit point of $\langle a_n \rangle$

Similarly, $\langle c_n \rangle = \left\langle (-1)^{2n+1} + \frac{1}{(2n+1)} \right\rangle$ is a subsequence of $\langle a_n \rangle$

and $\langle c_n \rangle \to -1$

$\Rightarrow$ -1 is a limit point of $\langle a_n \rangle$

Hence $\langle a_n \rangle$ is not convergent.

(c) Let $\langle a_n \rangle$ be a sequence of positive real numbers such then $a_n = \sqrt{a_{n-1}\, a_{n-2}}$ for $n > 2$ than $\langle a_n \rangle \to \left(a_1 a_2^2\right)^{1/3}$.

(d) Let $a_1 > 0, a_2 > 0$ and define $a_n = \frac{2a_{n-1}a_{n-2}}{a_{n-1} + a_{n-2}}, n > 2$, then $\langle a_n \rangle$ converges to $\frac{3a_1a_2}{2a_1 + a_2}$.

(e) The sequence $\langle a_n \rangle$, where $\langle a_n \rangle$ is defined as $a_n = 1 - \frac{1}{2} + \frac{1}{3} + \ldots + (-1)^{n+1}\frac{1}{n}$ is convergent.

Reason: We have $a_n = 1 - \frac{1}{2} + \frac{1}{3} + \ldots + (-1)^{n+1} \cdot \frac{1}{n}$

$$a_{2n} = 1 - \frac{1}{2} + \frac{1}{3} + \ldots + (-1)^{2n+1}\frac{1}{2n}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} + \ldots + \frac{(-1)}{2n}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n-1} - \frac{1}{2n} \qquad (2)$$

Let $\langle b_n \rangle = \langle a_{2n} \rangle$

Consider

$$b_n - b_{n-1} = \left(1-\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}\right)-$$

$$\left(1-\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n-3}-\frac{1}{2n-2}\right)$$

$$=\frac{1}{2n-1}-\frac{1}{2n}>0$$

$\Rightarrow b_n > b_{n-1}$

$\Rightarrow \langle b_n \rangle$ is monotonically increasing sequence.

$\rightarrow \langle a_{2n} \rangle$ is monotonically increasing sequence.

Now we show that $\langle a_{2n} \rangle$ is bouded above.

$$a_{2n} = 1-\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}$$

$$=1-\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\ldots-\frac{1}{2n-1}+\frac{1}{2n}\right)$$

$$=1-\left(\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots+\left(\frac{1}{2n-2}-\frac{1}{2n-1}\right)+\frac{1}{2n}\right)$$

$=1-\lambda,\ \lambda>0$ where

$$\lambda=\left(\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{2n-2}-\frac{1}{2n-1}\right)+\frac{1}{2n}\right)$$

$\Rightarrow \quad a_{2n} \le 1$

$\Rightarrow \langle a_{2n} \rangle$ is convergent $\forall\ n \in \mathbb{N}$.

Similarly we can show that $\langle a_{2n+1} \rangle$ is convergent.

Let $\lim\limits_{n\to\infty} a_{2n} = l_1$ and $\lim\limits_{n\to\infty} a_{2n+1} = l_2$

Now consider $a_{n+1} - a_n = (-1)^{n+1} \cdot \frac{1}{n}$

Taking $n \to \infty$ we have $l_1 - l_2 = 0 \Rightarrow l_1 = l_2$.

Since $\langle a_{2n+1} \rangle$, $\langle a_{2n} \rangle$ forms complementary sequences of $\langle a_n \rangle$ and converges to same limit

$\therefore \langle a_n \rangle$ is convergent.

(f) Let $\gamma_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n$

Then $\lim\limits_{n\to\infty} \gamma_n = \gamma$

3.5. Advanced Analysis of A Sequence

3.5.1. Limit Superior

Let $\langle a_n \rangle$ be a sequence of real numbers which is bounded above.

Define $b_1 = \sup\{a_1, a_2, \ldots a_{n+1}, \ldots\}$

$b_2 = \sup\{a_2, a_3, a_4, \ldots\}$

$\vdots$

$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$

Then $\langle b_n \rangle$ is defined which is monotonically decreasing.

Then limit superior of $\langle a_n \rangle$ is denoted by $\overline{\lim}\, a_n$ or $\lim\limits_{n\to\infty} \sup a_n$ and is defined as $\inf\{b_1, b_2, b_3, b_4, \ldots\}$.

If $\langle a_n \rangle$ is not bounded above. Then limit superior of $\langle a_n \rangle$ is defined as $\overline{\lim}\, a_n = \lim\limits_{n\to\infty} \sup a_n = +\infty$.

Example:

(a) Let $\langle a_n \rangle = \langle n \rangle$ Since it is not bounded above $\therefore\ \overline{\lim}\, a_n = +\infty$.

(b) Let $\langle a_n \rangle = \langle -n \rangle$

Clearly, $\langle a_n \rangle$ is bounded above

We have, $b_1 = \sup\{-1, -2, -3, \ldots\} = -1$

$b_2 = \sup\{-2, -3, -4, \ldots\} = -2$

$\vdots$

$b_n = \sup\{-n, -(n+1), \ldots\} = -n$

Now $\inf\{b_1, b_2, \ldots\} = -\infty$

$\therefore\ \overline{\lim}\, a_n = -\infty$

(c) Let $\langle a_n \rangle = (-1)^n . n$

Since $\langle a_n \rangle$ is not bounded above.

$\therefore\ \overline{\lim} \langle a_n \rangle = +\infty$.

(d) Let $\langle a_n \rangle = (-1)^n$

Clearly $\langle a_n \rangle$ is bounded sequence. $(\because -1 \le a_n \le 1\ \ \forall\, n \in \mathbb{N})$

We have, $b_1 = \sup\{-1, 1, -1, \ldots\} = 1$

$b_2 = \sup\{1, -1, \ldots\} = 1$

$\vdots$

$b_n = \sup\{(-1)^n, (-1)^{n+1}, \ldots\} = 1$

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Now $\inf\{b_1, b_2, \ldots\} = 1$

$\therefore\ \overline{\lim}\ a_n = 1.$

(e) Let $\langle a_n \rangle = \left\langle \frac{1}{n} \right\rangle$

Clear $\langle a_n \rangle$ is bounded sequence. $(\because 0 < a_n \le 1\ \ \forall\ n \in \mathbb{N})$

We have, $b_1 = 1$

$b_2 = \frac{1}{2}$

$\vdots$

$b_n = \frac{1}{n}$

Now $\inf\{b_1, b_2, \ldots, b_n, \ldots\} = 0$

$\therefore\ \overline{\lim}\ a_n = 0.$

3.5.2. Limit Inferior

Let $\langle a_n \rangle$ be a sequence of real numbers which is bounded below.

Define $c_1 = \inf\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\}$

$c_2 = \inf\{a_2, a_3, \ldots\}$

$\vdots$

$c_n = \inf\{a_n, a_{n+1}, \ldots\}$

Then $c_1 \le c_2 \le c_3 \le \ldots$ i.e., $\langle c_n \rangle$ is monotonically increasing sequence.

Now, limit inferior of $\langle a_n \rangle$ denoted by $\underline{\lim}\ a_n$ or $\lim\limits_{n\to\infty} \inf \langle a_n \rangle$ is defined to be $\sup\{c_1, c_2, c_3, \ldots\}$ and if $\langle a_n \rangle$ is not bounded below. Then we define $\underline{\lim}\, a_n = -\infty$

Examples:

(i) Let $\langle a_n \rangle = \langle n \rangle$, then $\langle a_n \rangle$ is bounded below by 1.

$c_1 = \inf\{1, 2, 3, \ldots\} = 1$

$c_2 = \inf\{2, 3, 4, \ldots\} = 2$

$\vdots$

$c_n = \inf\{n, n+1, \ldots\} = n$

and $\sup\{c_1, c_2, c_3, \ldots\} = +\infty$

$\therefore\ \underline{\lim}\, a_n = \infty.$

(ii) Let $\langle a_n \rangle = \langle -n \rangle$

Since $\langle a_n \rangle$ is not bounded below

$\therefore \ \underline{\lim}\, a_n = -\infty.$

(iii) Let $\langle a_n \rangle = \begin{cases} 4 & n = 1 \text{ or prime} \\ n & \text{else} \end{cases}$

Clearly $\langle a_n \rangle$ is bounded below by 4.

$c_1 = \inf\{4,4,4,4,4,6,4,\ldots\} = 4$

$c_2 = \inf\{4,4,4,4,6,4,8,\ldots\} = 4$

$\vdots$

$c_n = \inf\{a_n, a_{n+1}, \ldots\} = 4$

$\Rightarrow \ \sup\{c_1, c_2, \ldots\} = 4$

$\therefore \ \underline{\lim}\, a_n = 4.$

(iv) $\langle a_n \rangle = \begin{cases} -1 - \dfrac{1}{n} & \text{, when } n \text{ is even} \\ 1 + \dfrac{1}{n} & \text{, when } n \text{ is odd} \end{cases}$

Then $\overline{\lim}\, a_n = 1$ and $\underline{\lim}\, a_n = -1$

Note that $[-1,1] \cap R(U) = \phi$ (where $R(U)$ denote the range set of the sequence $\langle a_n \rangle$)

Note:

1. If α and β are limit inferior and limit superior of a sequence $\langle a_n \rangle$ respectively. Then $[\alpha, \beta]$ may not have any member of the sequence. (above example)
2. If the sequence is bounded. Then
   a. The Supremum of the set of limit points of the sequence is limit superior.
   b. The infimum of the set of limit points of the sequence is limit inferior.

Application: If $\langle a_n \rangle = (-1)^n$. Then limit superior and limit inferior of $\langle a_n \rangle$ are 1 and -1 respectively. Since set of limit points of $\langle a_n \rangle$ is $S = \{1, -1\}$ and $\sup S = 1$, $\inf S = -1$.

3.5.3. Convergent Sequence

A sequence $\langle a_n \rangle$ is said to be convergent iff limit superior is equal to the limit inferior and they exist finitely i.e. $\overline{\lim}\, a_n = \underline{\lim}\, a_n = l \, (l \in \mathbb{R})$, then l is called the limit of the sequence $\langle a_n \rangle$.

Example: Let $\langle a_n \rangle = \langle 2012 \rangle \ \forall \ n \in \mathbb{N}$, then $\langle a_n \rangle$ is convergent sequence $\because \underline{\lim}\, a_n = \overline{\lim}\, a_n = 2012.$

3.5.4. Divergent Sequence

A sequence $\langle a_n \rangle$ is said to be divergent If $\underline{\lim} a_n = \overline{\lim} a_n = \infty$ (infinite) Or $\underline{\lim} a_n = \overline{\lim} a_n = -\infty$ (infinite).

Examples:

1. Let $\langle a_n \rangle = \langle n \rangle$
2. Let $\langle a_n \rangle = \langle -n \rangle$

Note:

1. Divergent sequence is unbounded from one side.
2. Unbounded sequence may not be divergent sequence.

Example: Let $\langle a_n \rangle = (-1)^n . n$, then $\langle a_n \rangle$ is unbounded below and unbounded above both but it is not convergent sequence as $\underline{\lim} a_n = -\infty$ and $\overline{\lim} a_n = +\infty$.

3.5.5. Oscillatory Sequence

A sequence $\langle a_n \rangle$ is said to be oscillatory sequence if $\underline{\lim} a_n \neq \overline{\lim} a_n$.

3.5.6. Finitely Oscillatory Sequence

A sequence $\langle a_n \rangle$ is said to oscillate finitely if both $\overline{\lim} a_n$ and $\underline{\lim} a_n$ exist finitely and $\underline{\lim} a_n \neq \overline{\lim} a_n$.

Examples:

1. $\langle a_n \rangle = (-1)^n$
2. $\langle a_n \rangle = \begin{cases} -1 - \frac{1}{n} & n \text{ is even} \\ 1 + \frac{1}{n} & n \text{ is odd} \end{cases}$

3.5.7. Infinitely Oscillatory Sequence

A sequence $\langle a_n \rangle$ is said to be oscillate infinitely, if both $\underline{\lim} a_n$ and $\overline{\lim} a_n$ exist infinitely and $\underline{\lim} a_n \neq \overline{\lim} a_n$.

Examples: $\langle a_n \rangle = (-1)^n . n$

3.6. Properties on Limit Superior and Limit Inferior

1. Let $\langle a_n \rangle$ be a bounded sequence and $l \in \mathbb{R}$. Then $\lim\limits_{n \to \infty} \sup a_n = l$ iff for each $\varepsilon > 0$
   (i) $a_n > l - \varepsilon$ for infinitely many values of n.
   (ii) $a_n < l + \varepsilon$ for all except finitely many values of n.

2. Let $\langle a_n \rangle$ be a bounded sequence and $l \in \mathbb{R}$. Then $\lim\limits_{n\to\infty} \inf a_n = l$ iff for each $\in > 0$

   (i) $a_n < l + \varepsilon$ for infinitely many values of n.

   (ii) $a_n > l - \varepsilon$ for all except finitely many values of n.

3. Let $\langle a_n \rangle$ be a bounded sequence and $\lim\limits_{n\to\infty} \sup a_n = M$. Then

   (i) There is a subsequence of $\langle a_n \rangle$ that converges to M.

   (ii) The limit superior of no subsequence of $\langle a_n \rangle$ can exceed M.

   (iii) If a subsequence $\langle a_{n_k} \rangle$ of $\langle a_n \rangle$ converges to number l then $l \le M$.

4. Let $\langle a_n \rangle$ be a bounded sequence and $\lim\limits_{n\to\infty} \inf a_n = m$. Then

   (i) There is a subsequence of $\langle a_n \rangle$ that converges to m

   (ii) The limit inferior of no subsequence of $\langle a_n \rangle$ can be less than m.

   (iii) If a subsequence $\langle a_{n_k} \rangle$ of $\langle a_n \rangle$ converges to a number l then $l \ge m$.

5. Let $\langle a_n \rangle$ and $\langle b_n \rangle$ are two bounded sequences. Then

   (i) If $a_n \le b_n \;\forall\, n \in \mathbb{N} \Rightarrow \lim\limits_{n\to\infty} \sup a_n \le \lim\limits_{n\to\infty} \sup b_n$ and $\lim\limits_{n\to\infty} \inf a_n \le \lim\limits_{n\to\infty} \inf b_n$

   (ii) $\lim\limits_{n\to\infty} \sup(a_n + b_n) \le \lim\limits_{n\to\infty} \sup a_n + \lim\limits_{n\to\infty} \sup b_n$

   Equality hold when $\langle a_n \rangle$ is convergent.

   (iii) $\lim\limits_{n\to\infty} \inf(a_n + b_n) \ge \lim\limits_{x\to\infty} \inf a_n + \lim\limits_{n\to\infty} \inf b_n$

   Equality hold when $\langle a_n \rangle$ is convergent.

   (iv) $\lim\limits_{n\to\infty} \sup(a_n b_n) \le \lim\limits_{n\to\infty} \sup a_n . \lim\limits_{n\to\infty} \sup b_n$

   Equality hold when both $\langle a_n \rangle$, $\langle b_n \rangle$ are sequences of positive real numbers and $\langle a_n \rangle$ is convergent.

   (v) $\lim\limits_{n\to\infty} \inf(a_n b_n) \ge \lim\limits_{n\to\infty} \inf a_n . \lim\limits_{n\to\infty} \inf b_n$

   Equality hold when both $\langle a_n \rangle$, $\langle b_n \rangle$ are sequences of positive real numbers and $\langle a_n \rangle$ is convergent.

3.7. The Sequence of Natural Numbers

A sequence $\langle a_n \rangle$ is said to be sequence of natural number if its range set contains only natural numbers.

Results Based on Sequence of Natural Numbers

1. Every sequence of natural number has to be bounded below and has infimum in the range set.

   Reason: Obviously it is bounded below by 1.

2. If sequence of natural number has limit point p. Then this p has to be natural number & $a_n = p$ for infinite many value of n i.e., there exist subsequence $\langle a_{n_k} \rangle$ which is a constant sequence such that $a_{n_k} = p$ i.e., a sequence $\langle a_n \rangle$ of $\mathbb{N}$ has a limit point $\Leftrightarrow$ it has a constant subsequence.

   Reason: Since p is a limit point. So for $\in = \frac{1}{2}$, $(p - \in, p + \in)$ contains infinite member of $\mathbb{N} \Rightarrow a_n = p$ for infinite many values of $n \Rightarrow \exists$ a constant subsequence $\langle a_{n_k} \rangle = \langle p \rangle$ which converges to p.

   Conversely: If the sequence $\langle a_n \rangle$ of $\mathbb{N}$ has constant subsequence say $\langle a_{n_k} \rangle = \langle p \rangle$

   Then p is a limit point of $\langle a_n \rangle$.

3. Sequence of natural number is convergent iff it is eventually constant.

   Reason: Let $\langle a_n \rangle$ be a sequence of natural numbers such that $\lim\limits_{n \to \infty} a_n = l$.

   So, for $\in = \frac{1}{2}$ $\exists$ $m \in \mathbb{N}$ Such that

   $|a_n - l| < \frac{1}{2} \qquad \forall\ n \geq m$

   $\Rightarrow a_n = l \qquad \forall\ n \geq m.$

   $\Rightarrow \langle a_n \rangle$ is eventually sequence.

   Conversely: Suppose that $\langle a_n \rangle$ is eventually constant sequence. Then there exist $m \in \mathbb{N}$ such that $a_n = p \quad \forall\ n \geq m \ (p \in \mathbb{N})$

   $\therefore$ for any $\in > 0$

   $\therefore |a_n - p| < \in \qquad \forall\ n \geq m$

   $\therefore \Rightarrow \langle a_n \rangle$ is convergent and converges to p.

4. If a sequence of natural numbers is not divergent then it has constant subsequence.

Put Your Own Notes

Reason: Since sequence is not divergent then either it is convergent or Oscillates. If sequence oscillates then it is either Oscillates finitely or Oscillates infinitely .In both the cases, the sequence has limit point and hence it has a constant subsequence.

If sequence is convergent then there exist a constant subsequence.

3.8. Cauchy Sequence

A sequence $\langle a_n \rangle$ is said to be a **Cauchy sequence** if for any $\varepsilon > 0$, there exists a positive integer m such that $|a_n - a_m| < \varepsilon$, whenever $n \geq m$.

Example: The sequence $\langle a_n \rangle = \left\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots \right\rangle$ is a Cauchy sequence.

Reason: Let $\varepsilon > 0$ be given and if $n > m$, consider $|a_n - a_m| = \left|\frac{1}{n} - \frac{1}{m}\right| = \frac{m-n}{nm} < \frac{1}{n} < \varepsilon$, provided $m > \frac{1}{\varepsilon}$.

Let m be a positive integer greater than $1/\varepsilon$. Then $|a_n - a_m| < \in \quad \forall\, n \geq m$

Hence $\left\langle \frac{1}{n} \right\rangle$ is a Cauchy sequence.

Example: The sequence $\langle a_n \rangle = \left\langle \frac{n}{n+1} \right\rangle$ is a Cauchy sequence.

Reason: Let $\varepsilon > 0$ be given and if $n > m$, Consider

$$|a_n - a_m| = \left|\frac{n}{n+1} - \frac{m}{m+1}\right| = \frac{n-m}{(n+1)(m+1)} < \frac{1}{m}$$

$$\Rightarrow |a_n - a_m| < \frac{1}{m} < \varepsilon, \text{ provided } m > \frac{1}{\varepsilon}.$$

Let m be a positive integer greater than $1/\varepsilon$.

Then $|a_n - a_m| < \varepsilon$, for $n \geq m$.

Hence $\left\langle \frac{n}{n+1} \right\rangle$ is a Cauchy sequence.

Example: The sequence $\langle a_n \rangle = \langle n^2 \rangle$ is not a Cauchy sequence.

Reason: Let $n > m$, consider $|a_n - a_m| = n^2 - m^2 = (n-m)(n+m) > 2m > 1$, whatever m may be.

For $\varepsilon = \frac{1}{2}$, we see that there is no positive integer m such that $\left|n^2 - m^2\right| < \varepsilon$ for all $n \geq m$.

3.8.1. Algebra of Cauchy Sequences

Let $\langle a_n \rangle, \langle b_n \rangle$ are two Cauchy sequence. Then

1. $\langle a_n \rangle + \langle b_n \rangle = \langle a_n + b_n \rangle$ is a Cauchy sequence.
2. $\langle a_n \rangle - \langle b_n \rangle = \langle a_n - b_n \rangle$ is a Cauchy sequence.

3. $\langle a_n\rangle.\langle b_n\rangle=\langle a_n.b_n\rangle$ is a Cauchy sequence.

4. If each $a_n \neq 0\ \forall\ n\in\mathbb{N}$ and $\lim\limits_{n\to\infty} a_n \neq 0$ then $\left\langle \frac{1}{a_n}\right\rangle$ is Cauchy sequence.

Remark: If $\lim\limits_{n\to\infty} a_n = 0$ and each $a_n \neq 0$ where $\langle a_n\rangle$ is a Cauchy sequence. Then $\left\langle \frac{1}{a_n}\right\rangle$ may not be Cauchy sequence.

Example: Let $\langle a_n\rangle=\left\langle \frac{1}{n}\right\rangle$. Then $\langle a_n\rangle$ is a Cauchy sequence but $\left\langle \frac{1}{a_n}\right\rangle=\langle n\rangle$ is not a Cauchy sequence.

Cauchy's Criterion for Convergence

Statement: A sequence $\langle a_n\rangle$ of real numbers is convergent iff it is a Cauchy sequence.

3.8.2. Cauchy's General Principle of Convergence(CGPC)

A necessary and sufficient condition for a sequence $\langle a_n\rangle$ to be convergent is that to each $\in > 0$, there corresponds a positive integer m such that $|a_{n+p}-a_n| < \in\ \forall\ n\geq m, p\geq 0$.

Note: This principle is used where the limit l of sequence is not known no guess can be made of the same. This principle which involves only the terms of the sequence is useful for determining whether a sequence convergent or not.

Example:

1. The sequence $\langle a_n\rangle$, where $a_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$ is not convergent.

   Reason: Suppose that the given sequence is convergent.

   By Cauchy's General Principal of Convergence, for $\varepsilon=\frac{1}{2}$, there exists a positive integer m such that $|a_n-a_m|<\frac{1}{2}\ \forall n\geq m.$

   In particular for $n=2m$, $|a_{2m}-a_m|<\frac{1}{2}$. ...(1)

   Now $|a_{2m}-a_m| = \left|\left(1+\frac{1}{2}+\ldots+\frac{1}{m}+\frac{1}{m+1}+\ldots+\frac{1}{2m}\right)-\left(1+\frac{1}{2}+\ldots+\frac{1}{m}\right)\right|$

$$=\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{2m}$$

$$>\frac{1}{2m}+\frac{1}{2m}+\ldots+\frac{1}{2m}(m \text{ terms})=\frac{1}{2m}.m=\frac{1}{2}$$

$$\left(\because m+1<m+m \Rightarrow \frac{1}{m+1}>\frac{1}{2m} \text{ etc.}\right)$$

$\therefore \left|a_{2m} - a_m\right| > \frac{1}{2}$.

This contradicts (1).

Hence the given sequence is not convergent.

2. The sequence $\langle a_n \rangle$, where $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$ is a convergent sequence.

Reason: By Cauchy's criterion of convergence, a real sequence converges iff it is a Cauchy sequence. We shall show that the given sequence $\langle a_n \rangle$ is a Cauchy sequence, so that $\langle a_n \rangle$ is convergent.

For $n \geq m$, we have

$$\left|a_n - a_m\right| = \left|\left(1 + \frac{1}{1!} + \ldots + \frac{1}{m!} + \frac{1}{(m+1)!} + \frac{1}{(m+2)!} + \ldots + \frac{1}{n!}\right) - \left(1 + \frac{1}{1!} + \ldots + \frac{1}{m!}\right)\right|$$

$$= \frac{1}{(m+1)!} + \frac{1}{(m+2)!} + \ldots + \frac{1}{n!}$$

$$< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \ldots + \frac{1}{2^{n-1}} \qquad \left(\because n! \geq 2^{n-1} \forall n\right)$$

$$= \frac{1}{2^m}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{(n-m)-1}}\right]$$

$$= \frac{1}{2^m} \frac{\left[1 - \left(\frac{1}{2}\right)^{n-m}\right]}{1 - \left(\frac{1}{2}\right)} = \frac{1}{2^{m-1}} - \frac{1}{2^{n-1}} \leq \frac{1}{2^{m-1}}$$

$$\left(\because n \geq m \Rightarrow n-1 \geq m-1 \Rightarrow 2^{n-1} \geq 2^{m-1} \Rightarrow \frac{1}{2^{n-1}} \leq \frac{1}{2^{m-1}}\right)$$

For any $\varepsilon > 0, \left|a_n - a_m\right| < \varepsilon$, if $\frac{1}{2^{m-1}} < \varepsilon$ or $2^{m-1} > \frac{1}{\varepsilon}$

or $\quad (m-1)\log 2 > \log(1/\varepsilon)$ or $m > \left\{\log(1/\varepsilon)(\log 2)^{-1} + 1\right\} = N$, (say)

$\therefore \quad \left|a_n - a_m\right| < \varepsilon \ \forall \ n \geq m$, where m is a positive integer $\geq N$.

Hence $\langle a_n \rangle$ is a Cauchy sequence and so $\langle a_n \rangle$ is convergent.

3. Let $\langle a_n \rangle$ be a sequence of real numbers and for each $n \in \mathbb{N}$, define

$$s_n = a_1 + a_2 + \ldots + a_n, \; t_n = \left|a_1\right| + \left|a_2\right| + \ldots + \left|a_n\right| \qquad (1)$$

If $\langle t_n \rangle$ is a Cauchy sequence, then so is $\langle s_n \rangle$.

Reason: Since $\langle t_n \rangle$ is a Cauchy sequence, for any $\varepsilon > 0$, there exists a positive integer m such that $|t_n - t_m| < \varepsilon \ \forall \ n \geq m$

or $\left| \{|a_1| + \ldots + |a_m| + |a_{m+1}| + \ldots + |a_n|\} - \{|a_1| + |a_2| + \ldots + |a_m|\} \right| < \varepsilon \ \forall \ n \geq m$

or $\left| |a_{m+1}| + |a_{m+2}| + \ldots + |a_n| \right| < \varepsilon \ \forall \ n \geq m.$ (2)

Now $|s_n - s_m| = |a_{m+1} + a_{m+2} + \ldots + a_n|$, by (1)

$\leq |a_{m+1}| + |a_{m+2}| + \ldots + |a_n|$

$< \varepsilon \ \forall \ n \geq m$, by (2)

$\therefore \ |s_n - s_m| < \varepsilon \ \forall \ n \geq m$.

Hence $\langle s_n \rangle$ is a Cauchy sequence.

Some Important Theorems

1. **Cauchy's First Theorem on limits:** Let $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} \left(\frac{a_1 + a_2 + \ldots + a_n}{n} \right) = 0$

   In General: Let $\lim_{n\to\infty} a_n = l$. Then $\lim_{n\to\infty} \left(\frac{a_1 + a_2 + \ldots + a_n}{n} \right) = l$

   Example: $\lim_{n\to\infty} \frac{1}{n} \left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \right] = 0$.

   Reason: Let $a_k = \frac{1}{k}$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

   Since $\lim_{n\to\infty} a_n = 0$, so by Cauchy's first theorem on limits $\lim_{n\to\infty} \frac{1}{n} (a_1 + a_2 + \ldots + a_n) = 0$

   Hence $\lim_{n\to\infty} \frac{1}{n} \left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \right] = 0$

   Note: Converse of the above theorem need not be true.

   Example: $\langle a_n \rangle = \langle -1, 1, -1, 1, \ldots \rangle$ is not convergent and $a_1 + a_2 = 0, a_1 + a_2 + a_3 = -1, a_1 + a_2 + a_3 + a_4 = 0$, etc.

   $$\text{Now } \frac{a_1 + a_2 + \ldots + a_n}{n} = \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{1}{n}, & \text{if } n \text{ is odd} \end{cases}$$

   Hence $\lim_{n\to\infty} \left(\frac{a_1 + a_2 + \ldots + a_n}{n} \right) = 0$, but $\lim_{n\to\infty} a_n \neq 0$.

2. **Cauchy's Second Theorem on Limits:** If $\langle a_n \rangle$ converges to $l (\neq 0)$ and $a_n > 0$ then $\lim (a_1 a_2 \ldots a_n)^{1/n} = l$

Example: Let $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$. Then

$$\lim_{n\to\infty}\left[\frac{2}{1}\left(\frac{3}{2}\right)^2\left(\frac{4}{2}\right)^3\ldots\left(\frac{n+1}{n}\right)^n\right]^{1/n} = e.$$

Reason: Let $a_n = \left(1+\dfrac{1}{n}\right)^n$, then $a_n > 0\ \forall\ n\in\mathbb{N}$ and $\lim\limits_{n\to\infty} a_n = e > 0$.

By Cauchy's second theorem on limits, we have $\lim\limits_{n\to\infty}\left(a_1a_2a_3\ldots a_n\right)^{1/n} = e$.

Hence $\lim\limits_{n\to\infty}\left[\dfrac{2}{1}\left(\dfrac{3}{2}\right)^2\left(\dfrac{4}{3}\right)^3\ldots\left(\dfrac{n+1}{n}\right)^n\right]^{1/n} = e$, using (1).

3. If all the terms of a sequence $\langle a_n\rangle$ are positive and if $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}$ exist.Then $\lim\limits_{n\to\infty}(a_n)^{1/n}$ also exist and the two limits are equal. i.e., $\lim\limits_{n\to\infty}(a_n)^{1/n} = \lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}$, provided the later limit exist.

4. **Cesaro's Theorem:** Let $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$ where $a,b\in\mathbb{R}$,then $\lim\limits_{n\to\infty}\dfrac{a_1b_n + a_2b_{n-1}+\ldots+a_nb_1}{n} = a.b$

5. **Sandwich Theorem:** Let $\langle a_n\rangle,\langle b_n\rangle,\langle c_n\rangle$ are sequence of real number such that $a_n \le b_n \le c_n \quad \forall\ n\in\mathbb{N}$

   and $\lim\limits_{n\to\infty} a_n = l = \lim\limits_{n\to\infty} c_n$ then $\lim\limits_{n\to\infty} b_n = l$.

6. If $a_n \ge 0\ \forall\, n\in\mathbb{N}$ and $\lim\limits_{n\to\infty} a_n = l$, then $l \ge 0$.

7. If $\langle a_n\rangle,\langle b_n\rangle$ be two sequence such that $a_n \le b_n\ \forall\, n\in\mathbb{N}$ then $\lim a_n \le \lim b_n$

8. If $\langle a_n\rangle$ be a sequence of positive real numbers such that $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = l$, where $l<1$ then $\lim\limits_{n\to\infty} a_n = 0$.

   Application:

   (i) Let $\langle a_n\rangle = \left\langle 2^{-n}.n^2\right\rangle$, then $\lim\limits_{n\to\infty} 2^{(-n)}.n^2 = 0$

   Reason: We have $a_n = \dfrac{n^2}{2^n}$, $a_{n+1} = \dfrac{(n+1)^2}{2^{n+1}}$

   $$\therefore\ \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)^2}{2^{n+1}}.\frac{2^n}{n^2} = \lim_{n\to\infty}\frac{1}{2}\left(\frac{n+1}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right)^2 = \frac{1}{2} < 1$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{(-n)}.n^2 = 0$$

(ii) If $|x|<1$ and $m \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} \frac{m(m-1)(m-2)...(m-n+1)}{n!}x^n = 0$.

Reason: Let $a_n = \frac{m(m-1)...(m-n+1)}{n!}x^n$, so that $a_{n+1} = \frac{m(m-1)...(m-n+1)(m-n)}{(n+1)!}x^{n+1}$.

Now $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \left\{\frac{(m-n)}{(n+1)!}.n!.x\right\}$

$$= \lim_{n\to\infty} \left(\frac{m-n}{n+1}\right)x = \lim_{n\to\infty} \frac{\left(\frac{m}{n}\right)-1}{1+\left(\frac{1}{n}\right)}.x = -x$$

$\therefore \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = -x$, where $|-x| = |x| < 1$.

Hence $\lim\limits_{n\to\infty} a_n = 0$,

(iii) If $p > 0$, then $\lim\limits_{n\to\infty} \frac{n^k}{(1+p)^n} = 0$, k being a fixed real number.

Reason: Let $a_n = \frac{n^k}{(1+p)^n}$, so that $a_{n+1} = \frac{(n+1)^k}{(1+p)^{n+1}}$.

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^k.\frac{1}{1+p} = \left(1+\frac{1}{n}\right)^k.\frac{1}{1+p}$$

Now $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{(1+p)}.\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^k = \frac{1}{1+p}$,

Where $\frac{1}{1+p} < 1$ $(\because 1+p > 1 \text{ as } p > 0)$

Hence $\lim\limits_{n\to\infty} a_n = 0$.

9. If $\langle a_n \rangle$ be a sequence of positive real numbers such that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l > 1$, then $\langle a_n \rangle$ is not a bounded sequence and hence not convergent

10. Let $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$, and $\langle S_n\rangle$ & $\langle T_n\rangle$ are two sequences, where $S_n = \max\{a_n, b_n\}$ an $T_n = \min\{a_n, b_n\}$. Then the sequences $\langle S_n\rangle$ and $\langle T_n\rangle$ are convergent and $\lim\limits_{n\to\infty} S_n = \max\{a,b\}$, $\lim\limits_{n\to\infty} T_n = \min\{a,b\}$.

Reason: For any two real numbers x and y, we know

$$\max\{x,y\} = \frac{1}{2}(x+y) + \frac{1}{2}|x-y| \qquad (1)$$

$$\min\{x,y\} = \frac{1}{2}(x+y) - \frac{1}{2}|x-y| \qquad (2)$$

Taking $x = a_n$, $y = b_n$ in (1), we obtain $S_n = \max\{a_n, b_n\} = \frac{1}{2}(a_n + b_n) + \frac{1}{2}|a_n - b_n|$, using (1).

$$\therefore \lim_{n\to\infty} S_n = \frac{1}{2}\lim_{n\to\infty}(a_n + b_n) + \frac{1}{2}\lim_{n\to\infty}|a_n - b_n|$$

$$= \frac{1}{2}\left(\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n\right) + \frac{1}{2}\lim_{n\to\infty}|c_n|,\ c_n = a_n - b_n$$

$$= \frac{1}{2}(a+b) + \frac{1}{2}\lim_{n\to\infty}|c_n| \qquad (3)$$

We have $\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n = a - b$

$\therefore \lim\limits_{n\to\infty} c_n = a - b \Rightarrow \lim\limits_{n\to\infty}|c_n| = |a-b|$.

From (3), $\lim\limits_{n\to\infty} S_n = \frac{1}{2}(a+b) + \frac{1}{2}|a-b| = \max\{a,b\}$.

Similarly, $\lim\limits_{n\to\infty} T_n = \min\{a,b\}$

11. Consider a sequence $\langle a_n\rangle$, where $a_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$. Then $\langle a_n\rangle$ converges.

12. Let $\langle a_n\rangle$ be a sequence such that $a_n \to a$. Then $a_n^2 \to a^2$.

However, the converse may not be true.

For example, let $\langle a_n\rangle = \langle(-1)^n\rangle$. Then

$\langle a_n^2\rangle = \langle 1,1,1,\ldots\rangle$, which converges to 1.

but $\langle a_n\rangle$ does not converge to 1.

13. Let $\langle a_n\rangle$ be a sequence such that $a_n^2 \to a^2$, then $|a_n| \to |a|$ as $n \to \infty$.

CHAPTER 4

SERIES OF REAL NUMBERS

4.1. Definition

We know about arithmetic and geometric series etc. A series of n terms is denoted by the expression $u_1 + u_2 + ... + u_n$ or $\sum_{i=1}^{n} u_i$. If the series has no last term, then such series is called an infinite series.

4.1.1 Infinite Series:

Let $\langle a_n \rangle$ be a sequence of real numbers, then $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_{n+1} + ...$ i.e. infinite sum of the members of the sequence, is defined as series of real numbers. It is also denoted by $\sum a_n$.

Example:

(a) $\sum \frac{1}{n}$, here $\langle a_n \rangle = \left\langle \frac{1}{n} \right\rangle \quad \forall\, n \in \mathbb{N}$

(b) $\sum \sin\left(\frac{1}{n^2}\right)$, here $\langle a_n \rangle = \left\langle \sin\left(\frac{1}{n^2}\right) \right\rangle \quad \forall\, n \in \mathbb{N}$

4.1.2 Sequence of Partial Sums

Suppose $\sum a_n$ is an infinite series then we define a sequence $\langle S_n \rangle$ as follows:

$S_1 = a_1$

$S_2 = a_1 + a_2$

$\vdots \qquad \vdots$

$S_n = a_1 + a_2 + ... + a_n$, and so on.

The sequence $\langle S_n \rangle$ is called the sequence of partial sums of the series $\sum a_n$

4.1.3 Convergent Series

A series $\sum a_n$ is said to be convergent, if the sequence $\langle S_n \rangle$ of partial sums of $\sum a_n$ is convergent and if $\lim_{n\to\infty} S_n = S$, then S is called the sum of the series $\sum a_n$ and then we write it as $S = \sum_{n=1}^{\infty} a_n$.

4.1.4 Divergent Series

The series $\sum a_n$ is said to be divergent, if the sequence $\langle S_n \rangle$ of partial sums of $\sum a_n$ is divergent.

4.1.5 Oscillatory Series

The series $\sum a_n$ is said to oscillatory, if the sequence $\langle S_n \rangle$ of partial sums of $\sum a_n$ oscillates.

4.2. Necessary condition for convergence of a series

If the series $\sum a_n$ converge, then $\lim\limits_{n\to\infty} a_n = 0$.

Remark:

(a) Converse of the theorem need not be true.

Example: Let $\sum a_n = \sum \frac{1}{n}$, then $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ but $\sum \frac{1}{n}$ is not convergent.

(b) If $\lim\limits_{n\to\infty} a_n \neq 0$ then $\sum a_n$ cannot converge.

Examples:

(a) The series $1 + r + r^2 + \ldots (r > 0)$ converges if $r < 1$ and diverges if $r \geq 1$. **(Geometric Series)**

(b) The series $\sum (-1)^{n-1}$ oscillates.

Solution: We see that

$S_1 = 1,\ S_2 = 1 - 1 = 0,\ S_3 = 1 - 1 + 1 = 1,$

$S_4 = 1 - 1 + 1 - 1 = 0,\ S_5 = 1$ and so on.

$\therefore \langle S_n \rangle = \langle 1, 0, 1, 0, 1, 0, \ldots \rangle$, which oscillates.

Hence the given series oscillates.

(c) Test for convergence of the series :

(i) $\sum\limits_{n=1}^{\infty} (-1)^n n$,

(ii) $\sum\limits_{n=1}^{\infty} \sin\left(\frac{n\pi}{3}\right)$

Solution:

(i) We see that $\sum\limits_{n=1}^{\infty} (-1)^n n = -1 + 2 - 3 + 4 - 5 + 6 \ldots$

$\therefore S_1 = -1,\ S_2 = -1 + 2 = 1,\ S_3 = -1 + 2 - 3 = -2,$

$S_4 = -1 + 2 - 3 + 4 = 2,\ S_5 = -3,\ S_6 = 3$ etc.

$\Rightarrow \langle S_n \rangle = \langle -1, 1, -2, 2, -3, 3, \ldots \rangle$, which is not bounded.

Hence $\langle S_n \rangle$ is not convergent and so $\sum (-1)^n n$ is not convergent.

(ii) $\sum\limits_{n=1}^{\infty} \sin\frac{n\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + 0 - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + 0 + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + 0 \ldots$

The sequence $\langle S_n \rangle$ of partial sums of the above series is

$\left\langle \frac{\sqrt{3}}{2}, \sqrt{3}, \sqrt{3}, \frac{\sqrt{3}}{2}, 0, 0, \frac{\sqrt{3}}{2}, \sqrt{3}, \ldots \right\rangle$

Clearly, $\limsup S_n = \sqrt{3}$ and $\liminf S_n = 0$.

It follows that $\langle S_n \rangle$ is not convergent. Hence the given series is not convergent.

4.3. Telescopic Series

Let $\langle a_n \rangle$ be a sequence of real numbers.

Define $b_n = a_n - a_{n+1}$ and $c_n = a_{n+1} - a_n$

Then $\sum b_n$ and $\sum c_n$ are called telescopic series.

If $\langle S_n \rangle$ denote the sequence of partial sum of the series $\sum b_n$

Then $S_n = \sum_{r=1}^{n} b_r$

$= b_1 + b_2 + ... + b_n$

$= a_1 - a_2 + a_2 - a_3 + ... + a_n - a_{n+1}$

$= a_1 - a_{n+1}$

Similarly if $\langle t_n \rangle$ denote the sequence of partial sum of the series $\sum c_n$

Then $t_n = a_{n+1} - a_n$.

Thus $\langle S_n \rangle$ and $\langle t_n \rangle$ are convergent iff $\langle a_n \rangle$ is convergent.

Thus $\sum b_n$ and $\sum c_n$ are convergent iff $\langle a_n \rangle$ is convergent.

Example: The series $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ...$ is convergent.

Solution: We have,

$S_n = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)}$

$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + ... + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1}$

$\therefore \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1.$

Hence the given series is convergent.

4.4. Cauchy's General Principal of Convergence

A necessary and sufficient condition for a series $\sum a_n$ to converge is that for each $\varepsilon > 0$, there exists a positive integer m, such that $|a_{m+1} + a_{m+2} + ... + a_n| < \varepsilon$, for all $n \geq m$.

Pringsheim's Theorem: If a series $\sum u_n$ of positive monotonic decreasing terms converges then not only $u_n \to 0$ but also $nu_n \to 0$ as $n \to \infty$.

Reason: We know that for a convergent series, for any $\varepsilon > 0$, a positive integer N exists such that $\left|u_{m+1} + u_{m+2} + ... + u_{m+p}\right| < \frac{\varepsilon}{2}$, $\forall\, m \geq N, p \geq 1$

Let us choose $m + p = n > 2N$ and $m = \left[\frac{n}{2}\right]$, the greatest integer not greater than $\frac{n}{2}$ $u_{m+1} + u_{m+2} + ... + u_n < \frac{\varepsilon}{2}$

But $\sum u_n$ is positive monotonic decreasing so $(n-m)u_n < u_{m+1} + u_{m+2} + ... + u_n < \frac{\varepsilon}{2}$ or $\frac{1}{2} nu_n < \frac{1}{2}\varepsilon$ i.e. $nu_n < \varepsilon$, $\forall\, n \geq N$.

Hence $\lim nu_n = 0$.

Note: The condition $nu_n \to 0$ is only a necessary one not sufficient condition for the convergence of the present type of series. If nu_n does not tend to zero then the series $\sum u_n$ is certainly divergent, for example, the harmonic series $\sum \frac{1}{n}$ must diverge because it has positive monotonic decreasing terms and $n.\frac{1}{n}$ does not tend to zero. However, $nu_n \to 0$ does not imply anything as to the possible convergence of $\sum u_n$ for example, Abel's series $\sum \frac{1}{n \log n}$ diverges although it has positive monotonic decreasing terms and $nu_n \to 0$.

Example: The series $\sum \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots$ is not convergent.

Solution: We have $u_n = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1 \neq 0$$

Hence the given series is not convergent.

Example: The series $\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \ldots + \sqrt{\frac{n}{2(n+1)}} + \ldots$ is not convergent.

Solution: We have

$$u_n = \sqrt{\frac{n}{2(n+1)}} = \frac{1}{\sqrt{2}} \sqrt{\frac{1}{1+\frac{1}{n}}}.$$

$$\therefore \quad \lim_{n\to\infty} u_n = \frac{1}{\sqrt{2}} \neq 0.$$

Hence, the given series does not converge.

Example: Test for convergence of the series $\sum \cos\left(\frac{1}{n}\right)$.

Solution: $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \cos\frac{1}{n} = 1 \neq 0$

Hence the given series does not converge.

Example: The series $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{1/n}$ does not converge.

Solution: Let $u_n = \left(\frac{1}{n}\right)^{1/n}$, so that $\log u_n = \frac{1}{n} \log\left(\frac{1}{n}\right)$

$$= \log u_n = \frac{\log 1 - \log n}{n} = \frac{-\log n}{n}.$$

$\therefore \quad \lim_{n\to\infty} \log u_n = -\lim_{n\to\infty} \frac{\log n}{n}$, which is $\frac{\infty}{\infty}$ form

$= -\lim_{n\to\infty} \frac{1/n}{1}$ (by L'Hospital's Rule)

$$= -\lim_{n\to\infty}\frac{1}{n} = 0.$$

Now $\lim_{n\to\infty}\log u_n = 0 \Rightarrow \log\left(\lim_{n\to\infty} u_n\right) = 0$

$$\Rightarrow \lim_{n\to\infty} u_n = e^0 = 1 \neq 0.$$

Hence $\sum u_n$ is not convergent.

4.5. Series of Positive Real Numbers

Series with positive terms are the simplest and the most important type of series one comes across. The simplicity arises mainly from the fact that the sequence of its partial sums is monotonically increasing.

Let $\sum u_n$ be an infinite series of positive terms and $\{S_n\}$ be the sequence of its partial sums so that

$$S_n = u_1 + u_2 + ... + u_n \geq 0,\ \forall n$$

$$\therefore\ S_n - S_{n-1} = u_n \geq 0$$

$$\Rightarrow S_n \geq S_{n-1},\ \forall n > 1$$

Thus the sequence $\{S_n\}$ of partial sums of a series of positive terms is a monotonically increasing sequence.

Since a monotonic increasing sequence can either converge, or diverge to ∞, but cannot oscillate, therefore, there are only two possibilities for a positive term series, it may either converge or diverge to $+\infty$.

Remark:

(a) A positive term series converges iff the sequence of its partial sum is bounded above.

(b) The sequence of partial sum of a series with negative terms can be shown to be monotonic decreasing and hence a series with negative terms converges iff the sequence of its partial sum is bounded below.

(c) It may similarly be seen that a series of negative terms can either converge or diverge to $-\infty$.

(d) A series $\sum_{n=1}^{\infty} u_n$ whose terms are not necessarily positive may fail to be convergent even if the sequence $\langle S_n\rangle$ is bounded above.

For example consider $u_n = (-1)^n$ so that we have

$$S_n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

The sequence $\langle S_n\rangle$, even though bounded above, is obviously not convergent and as such the series is not convergent. The sequence $\langle S_n\rangle$ has two limit points viz. -1 and 0.

It should be seen that, in general, boundedness of the sequence of partial sums of a series is only a necessary but not a sufficient condition for the convergence of the series $\sum u_n$, and it is only for positive term series that this condition of boundedness of the sequence of its partial sums is as well a sufficient condition for the convergence of the corresponding series.

4.5.1. Test for Convergence of Positive Terms Series

1. **First Comparision Test:** If $\sum u_n$ and $\sum v_n$ are positive terms series, $k>0$ and $\exists\, m \in \mathbb{N}$ such that $u_n \le k v_n \;\; \forall \;\; n \ge m$

   (a) $\sum v_n$ converges $\Rightarrow \sum u_n$ converges.

   (b) $\sum u_n$ diverges $\Rightarrow \sum v_n$ diverges.

Example: Test for convergence the series $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots+\frac{1}{n^n}+\ldots$

Solution: Clearly, $n^n > 2^n$ for $n>2$.

$\therefore \frac{1}{n^n} < \frac{1}{2^n}$ for $n>2$

Since $\sum \frac{1}{2^n} = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots$ is a geometric series with common ratio $\frac{1}{2}<1$, so $\sum 1/2^n$ is convergent.

Hence by First Comparison Test $\sum \frac{1}{n^n}$ is convergent.

Example: The series $\sum e^{-n^2}$ converges.

Solution: We know $e^x > x$ for $x>0$. Then

$e^{n^2} > n^2 \Rightarrow \frac{1}{e^{n^2}} < \frac{1}{n^2} \Rightarrow e^{-n^2} < \frac{1}{n^2} \; \forall \, n$.

Since $\sum \frac{1}{n^2}$ converges, so by First Comparison Test, $\sum e^{-n^2}$ also converges.

Example: Test for convergence the series:

(a) $\sum_{n=2}^{\infty} \frac{1}{\log n}$,

(b) $\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$

Solution:

(a) We know $\log n < n$ for all $n \ge 2$.

$\therefore \;\; \frac{1}{\log n} > \frac{1}{n} \;\; \forall \;\; n \ge 2$ or $\frac{1}{n} < \frac{1}{\log n} \;\;\; \forall \;\; n \ge 2$.

Since $\sum \frac{1}{n}$ diverges, so by First Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\log n}$ diverges.

(b) We know $\frac{1}{n^2 \log n} < \frac{1}{n^2} \;\;\; \forall \;\; n \ge 3$.

Since $\sum \frac{1}{n^2}$ converges, so by First Comparison Test,

$\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$ converges.

Example: Test for convergence the series:

(a) $1+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}+...$

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$.

Solution:

(a) We know that $n! \geq 2^{n-1} \ \forall \ n \geq 2$.

$$\therefore \qquad \frac{1}{n!} \leq \frac{1}{2^{n-1}} \ \forall \ n \geq 2.$$

Now $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1+\frac{1}{2}+\frac{1}{2^2}+...$ being a geometric series with common ratio $\frac{1}{2}<1$ is convergent. Hence, by First Comparison Test, $\sum \frac{1}{n!}$ is convergent.

(b) We know $\frac{1}{\sqrt{n!}} \leq \frac{1}{2^{\frac{1}{2}(n-1)}} \ \forall \ n \geq 2$.

The series on the Right Hand Side being a geometric series with common ratio $\frac{1}{\sqrt{2}}<1$ is convergent. Hence $\sum \frac{1}{\sqrt{n!}}$ is convergent.

Example: Test for convergence the series $\sum_{n=1}^{\infty} \frac{1}{3^n+x}$, for all positive values of x.

Solution: Clearly $3^n + x > 3^n$, since $x > 0$

$$\Rightarrow \qquad \frac{1}{3^n+x} < \frac{1}{3^n} \ \forall \ n \in \mathbb{N}$$

Now $\sum \frac{1}{3^n}$ being a geometric series with common ratio $\frac{1}{3}<1$ is convergent. Hence, by First Comparison Test, the given series is convergent for all $x > 0$.

2. **Second Comparison Test:** If $\sum u_n$ and $\sum v_n$ are two positive term series such that $\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}} \ \forall \ n \geq m$ then

   (i) $\sum v_n$ converges $\Rightarrow \sum u_n$ converges.

   (ii) $\sum u_n$ diverges $\Rightarrow \sum v_n$ diverges.

3. **p Series Test:** The series $\sum \frac{1}{n^p}$ is convergent iff $p > 1$.

4. **Limit form Comparison Test:** Let $\sum u_n$ and $\sum v_n$ be two positive term series such that $\lim_{n\to\infty} \frac{u_n}{v_n} = l$, ($l$ is finite and non-zero). Then $\sum u_n$ and $\sum v_n$ converge or diverge together.

Remark: If $l = 0$ or $l = \infty$, then the conclusion of the above test may not hold good.

Example:

(a) Let $\sum u_n = \sum \frac{1}{n^2}$ and $\sum v_n = \sum \frac{1}{n}$.

Then $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{n}{n^2} = \lim\limits_{n\to\infty} \frac{1}{n} = 0$.

Thus $l = 0$ and in this case $\sum u_n$ converges and $\sum v_n$ diverges.

(b) Let $\sum u_n = \sum \frac{1}{n}$ and $\sum v_n = \sum \frac{1}{n^2}$.

Then $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{n^2}{n} = \lim\limits_{n\to\infty} n = \infty$.

Thus $l = \infty$ and in this case $\sum u_n$ diverges and $\sum v_n$ converges.

5. If $\sum a_n^2$ & $\sum b_n^2$ are convergent series of positive real numbers. Then $\sum \sqrt{a_n b_n}$ is convergent.

Proof: As $(a_n - b_n)^2 \ge 0 \quad \Rightarrow a_n^2 + b_n^2 - 2a_n b_n \ge 0$

$\Rightarrow \sum a_n^2 + \sum b_n^2 \ge \sum 2\sqrt{a_n b_n}$

$\Rightarrow \frac{\sum a_n^2 + \sum b_n^2}{2} \ge \sum \sqrt{a_n b_n}$

By comparison test, $\sum \sqrt{a_n b_n}$ is convergent. Hence proved.

6. If $\sum a_n$ is convergent. Then $\sum \frac{a_n}{n}$ is convergent.

Proof: As $\left(a_n - \frac{1}{n}\right)^2 \ge 0$

$\Rightarrow a_n^2 + \frac{1}{n^2} - \frac{2a_n}{n} \ge 0$

$\Rightarrow \sum a_n^2 + \sum \frac{1}{n^2} \ge \sum \frac{2a_n}{n}$

Since $\sum a_n^2$ and $\sum \frac{1}{n^2}$ are both convergent $\Rightarrow \sum a_n^2 + \sum \frac{1}{n^2}$ is convergent.

Then by comparison test $\sum \frac{2a_n}{n}$ is convergent

$\Rightarrow \sum \frac{a_n}{n}$ is convergent. Hence proved.

Example: Test each of the following series for convergence:

(a) $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots$

(b) $\frac{1}{\sqrt{1.2}} + \frac{1}{\sqrt{2.3}} + \frac{1}{\sqrt{3.4}} + \ldots$

Solution:

(a) $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots$

The n th term of this series is $u_n=\frac{2n-1}{n(n+1)(n+2)}$.

For large values of n, $u_n \sim \frac{2n}{n.n.n}=\frac{2}{n^2}$.

Let $v_n=\frac{1}{n^2}$. Then $\frac{u_n}{v_n}=\frac{(2n-1)n^2}{n(n+1)(n+2)}$

$=\left(\frac{n}{n+1}\right)\left(\frac{2n-1}{n+2}\right)=\frac{1}{1+(1/n)}.\frac{2-(1/n)}{1+(2/n)}$.

Now $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=2\neq 0$ and finite.

So, by limit form comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n=\sum\frac{1}{n^2}$ converges, so $\sum u_n$ converges.

(b) $\frac{1}{\sqrt{1.2}}+\frac{1}{\sqrt{2.3}}+\frac{1}{\sqrt{3.4}}+\ldots$

The n th term is $u_n=\frac{1}{\sqrt{n(n+1)}}\sim\frac{1}{\sqrt{n.n}}\sim\frac{1}{n}$

Let $v_n=\frac{1}{n}$. Then $\frac{u_n}{v_n}=\sqrt{\frac{n}{n+1}}=\frac{1}{\sqrt{1+(1/n)}}$.

Now $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=1\neq 0$ and finite.

So, $\sum u_n$ and $\sum v_n$ converges or diverge together by limit form comparison test

Since $\sum v_n=\sum\frac{1}{n}$ diverges, so $\sum u_n$ diverges.

Example: Test for convergence the series $\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\frac{1}{\sqrt{4}+\sqrt{5}}+\ldots$

Solution: We have $u_n=\frac{1}{\sqrt{n+1}+\sqrt{n+2}}\sim\frac{1}{\sqrt{n}+\sqrt{n}}=\frac{1}{2\sqrt{n}}$.

Let $v_n=\frac{1}{\sqrt{n}}$. Then $\frac{u_n}{v_n}=\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}}$.

Now, $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=\lim\limits_{n\to\infty}\frac{1}{\sqrt{1+(1/n)}+\sqrt{1+(2/n)}}=\frac{1}{\sqrt{2}}\neq 0$ and finite

So $\sum u_n$ and $\sum v_n$ converge or diverge together by limit form comparison test,

Since $\sum v_n=\sum\frac{1}{\sqrt{n}}$ diverge, so $\sum u_n$ diverges.

Put Your Own Notes

Example: Test for convergence the series $\sum u_n = \sum \frac{\sqrt{2n-1}}{(2n+2)(2n+4)}$

Solution: We have, $u_n = \frac{\sqrt{2n-1}}{(2n+2)(2n+4)} \sim \frac{\sqrt{2n}}{2n.2n}$.

Let $v_n = \frac{\sqrt{n}}{n.n} = \frac{1}{n^{3/2}}$.

Then $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \sqrt{\frac{2n-1}{n}}\left(\frac{n}{2n+2}\right)\left(\frac{n}{2n+4}\right)$

$= \lim\limits_{n\to\infty} \sqrt{2-\frac{1}{n}}\,\frac{1}{\left(2+\frac{2}{n}\right)}\,\frac{1}{\left(2+\frac{4}{n}\right)}$

$= \frac{\sqrt{2}}{2.2} = \frac{1}{2\sqrt{2}} \neq 0$ and finite

So $\sum u_n$ and $\sum v_n$ converge or diverge together.(By limit form comparison test)

Since $\sum v_n = \sum \frac{1}{n^{3/2}}$ converges, so the given series converges.

Example: Test for converges the series $\frac{1}{3.7} + \frac{1}{4.9} + \frac{1}{5.11} + \frac{1}{6.13} + \ldots$

Solution: We have $u_n = \frac{1}{(n+2)(2n+5)} \sim \frac{1}{n.2n} = \frac{1}{2n^2}$

Let $v_n = \frac{1}{n^2}$.

Then $\frac{u_n}{v_n} = \frac{n}{n+2}.\frac{n}{2n+5} = \frac{1}{1+(2/n)}.\frac{1}{2+(5/n)}$.

$\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2} \neq 0$ and finite.

So, by limit form comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n = \sum 1/n^2$ converges, so the given series $\sum u_n$ also converges.

Example: Test for convergence the series whose n th term is $u_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

Solution: We have $u_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n} = \frac{\left(\sqrt{n+1}-\sqrt{n-1}\right)\left(\sqrt{n+1}+\sqrt{n-1}\right)}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)}$

(Rationalize)

$= \frac{2}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)} \sim \frac{2}{n\left(\sqrt{n}+\sqrt{n}\right)} = \frac{1}{n\sqrt{n}}$

Let $v_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$, so that $\sum v_n$ converges.

Now $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=\lim\limits_{n\to\infty}\frac{2\sqrt{n}}{\left(\sqrt{n+1}+\sqrt{n-1}\right)}$

$\lim\limits_{n\to\infty}\frac{2}{\sqrt{1+(1/n)}+\sqrt{1-(1/n)}}=1\neq 0$ and finite

So $\sum u_n$ and $\sum v_n$ converge or diverge together by limit form comparison test.

Since $\sum v_n$ converges, so the given series $\sum u_n$ also converges.

Example: The series $\sum\frac{1}{\sqrt{n}+\sqrt{n+1}}$ diverges.

Solution: Take $v_n=\frac{1}{\sqrt{n}}$, then $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=\frac{1}{2}\neq 0$ and finite.

By limit form comparison test, the given series diverges.

Example: Test for convergence the series $\sum u_n=\sum\limits_{n=1}^{\infty}\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$.

Solution: $u_n=\sqrt{n^3+1}-\sqrt{n^3}=\frac{\left(\sqrt{n^3+1}-\sqrt{n^3}\right)\left(\sqrt{n^3+1}+\sqrt{n^3}\right)}{\sqrt{n^3+1}+\sqrt{n^3}}$

$=\frac{\left(n^3+1\right)-n^3}{\sqrt{n^3+1}+\sqrt{n^3}}=\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}\sim\frac{1}{2\sqrt{n^3}}$

Let $v_n=\frac{1}{\sqrt{n^3}}=\frac{1}{n^{3/2}}$, so that $\sum v_n$ converges.

Now $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=\lim\limits_{n\to\infty}\frac{\sqrt{n^3}}{\sqrt{n^3+1}+\sqrt{n^3}}$

$=\lim\limits_{n\to\infty}\frac{1}{\sqrt{1+(1/n)^3}+1}=\frac{1}{2}\neq 0$ and finite.

So, by limit form comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together.

Since, $\sum v_n$ converges, so the given series $\sum u_n$ converges.

Example: Test for convergence the series whose n th term is $u_n=\left\{\left(n^3+1\right)^{1/3}-n\right\}$.

Solution: Let $u_n=\left(n^3+1\right)^{1/3}-n-n\left[\left(1+\frac{1}{n^3}\right)^{1/3}-1\right]$

$=n\left[\left(1+\frac{1}{3n^3}-\frac{1}{9n^6}+\ldots\right)-1\right]=\frac{1}{3n^2}-\frac{1}{9n^5}+\ldots$

Let $v_n=\frac{1}{n^2}$, so that $\sum v_n$ converges.

Now, $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=\frac{1}{3}\neq 0$ and finite.

So, by limit form comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n$ converges, so $\sum u_n$ converges.

Example: Test for convergence the series $\sum u_n = \sum \frac{n+1}{n^p},\quad p\in\mathbb{R}$

Solution: Let $u_n=\frac{n+1}{n^p}$ and $v_n=\frac{n}{n^p}=\frac{1}{n^{p-1}}$.

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{n+1}{n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1\neq 0 \text{ and finite.}$$

So, by limit form comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together by limit form comparison test.

Now, $\sum v_n=\sum\frac{1}{n^{p-1}}$ converges, if $p-1>1$ *i.e.*, $p>2$ and diverges if $p-1\leq 1$ *i.e.*, $p\leq 2$.

Hence $\sum\frac{n+1}{n^p}$ converges if $p>2$ and diverges if $p\leq 2$.

Example: Test the convergence of the series $\sum\frac{1}{n^{1+(1/n)}}$.

Solution: We have $u_n=\frac{1}{n^{1+(1/n)}}$. Let $v_n=\frac{1}{n}$.

Then $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=\lim\limits_{n\to\infty}\frac{1}{n^{1/n}}=1$.

Thus the two series $\sum u_n$ and $\sum v_n$ converge or diverge together by limit form comparison test.

Since $\sum v_n=\sum 1/n$ diverges, so the given series $\sum u_n$ diverges.

Example: Test for convergence the series $\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\ldots$

Solution: The n th term of the given series is $u_n=\frac{\sqrt{n+1}-1}{(n+2)^3-1}$.

Let $v_n=\frac{\sqrt{n}}{n^3}=\frac{1}{n^{5/2}}$

$$\text{Now, }\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left(\sqrt{\frac{n+1}{n}}-\frac{1}{\sqrt{n}}\right)\lim_{n\to\infty}\frac{1}{\left[\left(\left(\frac{n+2}{n}\right)^3-\frac{1}{n^3}\right)\right]}$$

$$=\lim_{n\to\infty}\sqrt{1+\left(\frac{1}{n}\right)}.\lim_{n\to\infty}\frac{1}{\left[1+\left(\frac{2}{n}\right)\right]^3}=1.$$

So $\sum u_n$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n=\sum\frac{1}{n^{5/2}}$ converges, so the given series converges.

7. If $\sum a_n$ is a convergent series of positive terms then $\sum \frac{a_n}{1+a_n}$ is convergent.

Reason: Since $\sum a_n$ is convergent so $\lim\limits_{n\to\infty} a_n = 0$.

Let $b_n = \frac{a_n}{1+a_n}\ \forall n$. Then $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = \lim\limits_{n\to\infty}(1+a_n) = 1+0 = 1$

Hence, $\sum a_n$ and $\sum b_n$ converge or diverge together by limit form comparison test. Since $\sum a_n$ is convergent, so $\sum b_n = \sum \frac{a_n}{1+a_n}$ is convergent.

Remark: If $\sum a_n$ is convergent $\Rightarrow \sum \frac{a_n}{1-a_n}$ is convergent $(a_n \ge 0 \text{ and } a_n \ne 1\ \forall\, n \in \mathbb{N})$

Take $b_n = \frac{a_n}{1-a_n}$ then $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = \lim\limits_{n\to\infty}(1-a_n) = 1 \ne 0$ so, by limit form comparison test, $\sum a_n$ & $\sum b_n$ converges & diverges together. Since $\sum a_n$ is convergent, so $\sum \frac{1}{1-a_n}$ is convergent.

8. **D' Alembert's Ratio Test :** Let $\sum a_n$ be a positive term series such that $\lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}} = l$. Then

(i) $\sum a_n$ converges if $l > 1$

(ii) $\sum a_n$ diverges if $l < 1$

(iii) Test fails if $l = 1$

Example: Test for convergence the series $\frac{1}{2} + \frac{1}{2.2^2} + \frac{1}{3.2^3} + \frac{1}{4.2^4} + \ldots$

Solution: We have $u_n = \frac{1}{n.2^n}, u_{n+1} = \frac{1}{(n+1)2^{n+1}}$.

and $\lim\limits_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim\limits_{n\to\infty}\frac{2(n+1)}{n} = 2\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right) = 2 > 1$.

By Ratio Test, the given series converges.

Example: Test for convergence the series $\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \ldots$

Solution: We have

$$u_n = \left[\frac{1.2.3\ldots n}{3.5.7\ldots(2n+1)}\right]^2,\ u_{n+1} = \left[\frac{1.2.3\ldots n(n+1)}{3.5.7\ldots(2n+1)(2n+3)}\right]^2$$

and $\lim\limits_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim\limits_{n\to\infty}\left[\frac{2n+3}{n+1}\right]^2 = \lim\limits_{n\to\infty}\left(\frac{2+\frac{3}{n}}{1+\frac{1}{n}}\right) = 4 > 1$.

By Ratio Test, the given series converges.

Put Your Own Notes

Example: Test for convergence the series $\sum_{n=1}^{\infty} \frac{1.2.3 \ldots n}{7.10 \ldots (3n+4)}$

Solution: We have $u_n \frac{1.2.3 \ldots n}{7.10 \ldots (3n+4)}, u_{n+1} = \frac{1.2.3 \ldots n(n+1)}{7.10 \ldots (3n+4)\{3(n+1)+4\}}$.

and $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{3n+7}{n+1} = \lim_{n\to\infty} \frac{3+\frac{7}{n}}{1+\frac{1}{n}} = 3 > 1$

By Ratio Test, the given series converges.

Example: Test for convergence the series $\frac{1^2.2^2}{1!} + \frac{2^2.3^2}{2!} + \frac{3^2.4^2}{3!} + \ldots$

Solution: We have $u_n = \frac{n^2(n+1)^2}{n!}, u_{n+1} = \frac{(n+1)^2(n+2)^2}{(n+1)!}$

and $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+1)!}{n!}\left(\frac{n}{n+2}\right)^2$

$= \lim_{n\to\infty}(n+1). \lim_{n\to\infty}\left(\frac{1}{1+\frac{2}{n}}\right)^2 = \infty$

Hence the series converges.

Example: Test for convergence the series $\sum_{n=1}^{\infty} \frac{r^n}{n!}$, where r is any positive number.

Solution: We have $u_n = \frac{r^n}{n!}, u_{n+1} = \frac{r^{n+1}}{(n+1)!}$.

Now $\frac{u_n}{u_{n+1}} = \frac{1}{r}\frac{(n+1)!}{n!}$,

and $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{r} \lim_{n\to\infty}(n+1) = \infty$ $[\because r > 0]$

By Ratio Test, the given series converges.

Example: Test for convergence the series whose n th term is $\frac{r^n}{n^n}, r > 0$

Solution: We have $u_n = \frac{r^n}{n^n}, u_{n+1} = \frac{r^{n+1}}{(n+1)^{n+1}}$.

$\frac{u_n}{u_{n+1}} = \frac{1}{r}\frac{(n+1)^{n+1}}{n^n} = \frac{1}{r}\left(\frac{n+1}{n}\right)^n (n+1)$

$= \frac{1}{r}\left(1+\frac{1}{n}\right)^n (n+1)$.

Now $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{r}.e.\infty = \infty$

By Ratio Test, the given series converges.

Example: Test for convergence the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$.

Solution: We have $u_n=\frac{n!}{n^n},\ u_{n+1}=\frac{(n+1)!}{(n+1)^{n+1}}$.

Now $\frac{u_n}{u_{n+1}}=\frac{n!}{(n+1)!}\frac{(n+1)^{n+1}}{n^n}$

$=\frac{1}{(n+1)}\left(\frac{n+1}{n}\right)^n(n+1)=\left(1+\frac{1}{n}\right)^n$.

$\lim\limits_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e=2.718>1$.

By Ratio Test, the given series converges.

Example: Test for convergence the series $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$ for all positive values of x.

Solution: $u_n=\frac{x^{2n-1}}{(2n-1)!},\ u_{n+1}=\frac{x^{2n+1}}{(2n+1)!}$.

Now $\frac{u_n}{u_{n+1}}=\frac{1}{x^2}\frac{(2n+1)!}{(2n-1)!}$

$=\frac{1}{x^2}\frac{(2n+1).2n.(2n-1)!}{(2n-1)!}$

$\lim\limits_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim\limits_{n\to\infty}\frac{2n(2n+1)}{x^2}=\infty$.

Hence, by Ratio Test, $\sum u_n$ converges.

9. **Rabbe's Test:** Let $\sum a_n$ be a positive term series such that $\lim\limits_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=l$. Then

(i) $\sum a_n$ converges if $l>1$

(ii) $\sum a_n$ diverges if $l<1$

(iii) Test fails if $l=1$

Example: Test for convergence the series $\sum_{n=1}^{\infty}\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}.\frac{1}{n}$.

Solution: We have $u_n=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}.\frac{1}{n}$,

$u_{n+1}=\frac{1.3.5\ldots(2n-1)(2n+1)}{2.4.6\ldots 2n(2n+2)}.\frac{1}{n+1}$

$\frac{u_n}{u_{n+1}}=\frac{(2n+2)(n+1)}{(2n+1)n}=\left(1+\frac{1}{n}\right)\frac{(1+1/n)}{(1+1/2n)}$

Now $\lim\limits_{n\to\infty}\dfrac{u_n}{u_{n+1}}=1$ means that the Ratio test fails.

So, we apply Rabbe's Test. We have

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\frac{(n+1)(2n+2)}{n(2n+1)}-1\right]=\frac{3n+2}{2n+1}=\frac{3+\frac{2}{n}}{2+\frac{1}{n}}.$$

and $\lim\limits_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}}-1\right)=\dfrac{3}{2}>1$.

Hence, by Raabe's test, the given series converges.

Example: Test for convergence the series $\sum\limits_{n=1}^{\infty}\dfrac{1.3.5...(2n-1)}{2.4.6...2n}.\dfrac{4n+1}{2n+3}$,

Solution: We have $u_n=\dfrac{1.3...(2n-1)(4n+1)}{2.4...2n(2n+3)}, u_{n+1}=\dfrac{1.3...(2n+1).(4n+5)}{2.4...(2n+2).(2n+5)}$.

$$\frac{u_n}{u_{n+1}}=\frac{2n+2}{2n+1}.\frac{2n+5}{4n+5}.\frac{2n+3}{4n+1}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{2.2.2}{2.4.4}=\frac{1}{4}<1.$$

By Raabe's test, $\sum u_n$ diverges.

10. **Cauchy's nth Root Test :** Let $\sum a_n$ be a positive term series such that $\lim\limits_{n\to\infty}(a_n)^{1/n}=l$. Then

(i) $\sum a_n$ converges if $l<1$

(ii) $\sum a_n$ diverges if $l>1$

(iii) Test fails if $l=1$

Example: Test for convergence the series whose n^{th} term is $u_n=\dfrac{n^{n^2}}{(n+1)^{n^2}}$.

Solution: We have $u_n=\left[\dfrac{n}{n+1}\right]^{n^2}$

$$\Rightarrow (u_n)^{1/n}=\left(\frac{n}{n+1}\right)^n=\frac{1}{(1+1/n)^n}.$$

and $\lim\limits_{n\to\infty}(u_n)^{1/n}=\dfrac{1}{\lim\limits_{n\to\infty}(1+1/n)^n}=\dfrac{1}{e}<1. (e=2.718)$.

Hence, by Cauchy's root test, the given series is convergent.

Example: The series $\sum\limits_{n=1}^{\infty}\left(n^{1/n}-1\right)^n$ converges.

Solution: Let $u_n=\left(n^{1/n}-1\right)^n$, so that $(u_n)^{1/n}=n^{1/n}-1$.

Now $\lim\limits_{n\to\infty}(u_n)^{1/n}=\lim\limits_{n\to\infty}n^{1/n}-1=1-1=0<1$.

Hence, by Cauchy's root test, the given series converges.

Put Your Own Notes

Example: Test for the convergence the series whose n^{th} term is $u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Solution: We have $u_n = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}}$.

Then $(u_n)^{1/n} = \frac{1}{\lim\limits_{n\to\infty}\left(1+1/\sqrt{n}\right)^{\sqrt{n}}} = \frac{1}{e} < 1$ $(\because e = 2.7182)$

Hence, by Cauchy's root test, the given series converges.

Example: Test the convergence of the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots$$

Solution: The n th term of the given series is $u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$.

Now, $(u_n)^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1}$

$= \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$

$= \left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^{n} - 1\right]^{-1}$.

$\lim\limits_{n\to\infty}(u_n)^{1/n} = 1.(e-1)^{-1} = \frac{1}{e-1} < 1 (e = 2.718)$.

Hence, by Cauchy's root test, the given series converges.

Example: Test for convergence the series whose n th term is $2^{-n-(-1)^n}$.

Solution: Let $u_n = 2^{-n-(-1)^n} = 2^{-\{n+(-1)^n\}}$.

and $(u_n)^{1/n} = \frac{1}{\left[2^{n+(-1)^n}\right]^{1/n}} = \frac{1}{2^{\frac{n}{n}}.2^{\frac{(-1)^n}{n}}}$

Now $\lim\limits_{n\to\infty}(u_n)^{1/n} = \frac{1}{2}.\frac{1}{\lim\limits_{n\to\infty} 2^{\frac{(-1)^n}{n}}} = \frac{1}{2} < 1.$ $\left[\because \lim\limits_{n\to\infty}\frac{(-1)^n}{n} = 0\right]$

Hence, by Cauchy's root test, the given series converges.

Some Important Definitions

1. The infinite integral $\int_1^{\infty} f(x)\,dx$ is said to be convergent (divergent), if $\lim\limits_{t\to\infty}\int_1^{t} f(x)\,dx$ is finite (infinite).

2. Let $f(x)$ be a real valued function with domain $[1,\infty[$. Then the function $f(x)$ is said to be **non-negative,** if $f(x)\geq 0\ \forall\ x\geq 1$.

3. The function $f(x)$ is said to be **monotonically decreasing,** if $x\leq y\Rightarrow f(x)\geq f(y)\ \forall\ x,y\in[1,\infty[$.

   Example: Let $f(x)=1/x^2$. Then $f(x)$ is non-negative and monotonically decreasing $\forall\ x\geq 1$.

11. **Cauchy's Integral Test:** If $u(x)$ is a non-negative, monotonically decreasing and integrable function such that $u(n)=u_n\ \ \forall n\in\mathbb{N}$, then the series $\sum_{n=1}^{\infty}u_n$ is convergent if and only if $\int_1^{\infty}u(x)\,dx$ is convergent.

Example: The series $\sum_{n=1}^{\infty}\frac{1}{n^p}(p>0)$ is convergent if $p>1$ and divergent if $p\leq 1$.

Example: The series $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^p}$ is convergent if $p>1$ and divergent if $0<p\leq 1$.

Example: examine the convergence of the following series $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$.

Solution: Let $u(x)=\frac{1}{x^2+x}$, so that $u(n)=u_n\ \forall\ n\in\mathbb{N}$.

For $x\geq 1$, $u(x)$ is non-negative, monotonically decreasing and integrable function. Now

$$\int_1^t\frac{dx}{x^2+x}=\int_1^t\frac{dx}{x(x+1)}=\int_1^t\left(\frac{1}{x}-\frac{1}{x+1}\right)dx$$

$$=\left|\log x-\log(x+1)\right|_1^t=\left|\log\frac{x}{1+x}\right|_1^t$$

$$=\log\frac{t}{1+t}-\log\frac{1}{2}=\log\frac{1}{1+(1/t)}-\log\frac{1}{2}.$$

$\therefore\ \lim_{t\to\infty}\int_1^t\frac{dx}{x^2+x}=\log 1+\log 2=\log 2$, which is finite.

Thus $\int_1^{\infty}u(x)dx$ is a convergent and so by Cauchy's Integral Test, the given series is convergent.

Example: The series $\sum_{n=3}^{\infty}\frac{1}{n\log n(\log\log n)^p}$, $(p>0)$ converges $p>1$ and diverges if $0<p\leq 1$.

4.6. Alternating Series

A series of the form $u_1 - u_2 + u_3 \ldots$ where $u_n > 0 \ \forall\, n \in \mathbb{N}$ is called an alternating series and is denoted by $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$.

Example:

(a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

(b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$

(c) $\sum_{n=2}^{\infty}\frac{(-1)^n}{\log n} = \frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots$

4.6.1 Leibnitz's Test for Alternating Series:

Leibnitz's Test for Alternating Series: If an alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ satisfies

(i) $u_{n+1} \le u_n \ \forall\, n$

(ii) $\lim_{n\to\infty} u_n = 0$

Then, the series $\sum(-1)^{n-1}u_n$ converges.

Remark: The alternating series $\sum(-1)^{n-1}u_n$ will not be convergent if either $u_{n+1} \nleq u_n,\ \forall\, n$ or $\lim_{n\to\infty} u_n \ne 0$.

Example: The series $1 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots$ is not convergent, as $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 1 \ne 0$.

Example: Test for convergence the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

Solution: We see that $\frac{1}{2} < 1, \frac{1}{3} < \frac{1}{2}, \frac{1}{4} < \frac{1}{3}$ and so on. Thus

(a) $u_{n+1} < u_n \ \forall\, n$

(b) $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n} = 0$

Hence, by Leibnitz's Test, the given series is convergent.

4.7. Absolute Convergence

If $\sum a_n$ is a series of real numbers such that $\sum|a_n|$ is a convergent series. Then $\sum a_n$ is called an absolutely convergent series.

Example:

(a) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ is absolute convergent series.

(b) $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n^2}\right)$ is absolute convergent series

(c) $\sum\frac{(-1)^{n+1}}{n}$ is not absolutely convergent series.

Result: Let $\sum u_n$ be absolutely convergent series. Then $\sum u_n$ is also convergent. The converse of this is not true i.e., a convergent series may not be absolutely convergent.

Example: Consider the series $\sum u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

We have seen that the given series is convergent, by Leibnitz's Test but $\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \ldots = \sum \frac{1}{n}$ is not convergent i.e., $\sum u_n$ is not absolutely convergent.

4.7.1 Conditional Convergence

A series $\sum u_n$ is said to be conditionally convergent, if

(i) $\sum u_n$ is convergent, and

(ii) $\sum u_n$ is not absolutely convergent.

Example: Test for convergence and absolute convergence the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots$ for $p > 0$.

Solution: Since $p > 0$ and $(n+1)^p > n^p$, so

$$\frac{1}{(n+1)^p} < \frac{1}{n^p} \quad \Rightarrow u_{n+1} < n_n \quad \forall\, n$$

Also, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^p} = 0$, since $p > 0$.

Be Leibnitz's Test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ is convergent.

Now, $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^p} \right| = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$. Hence the given series is absolutely convergent if $p > 1$ and conditionally convergent if $0 < p \le 1$.

Example: Test for convergence and absolute convergence the series $\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{7}} + \ldots$

Solution: We have $u_{n+1} < u_n$ for each n and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{2n-1}} = 0.$$

Thus $\sum u_n$ is convergent, by Leibnitz's test.

Now we show $\sum u_n$ is not absolutely convergent. We have $|u_n| = \frac{1}{\sqrt{2n-1}}$.

Let $v_n = \frac{1}{\sqrt{n}} \Rightarrow \lim_{n\to\infty} \frac{|u_n|}{v_n} = \lim_{n\to\infty} \sqrt{\frac{n}{2n-1}} = \frac{1}{\sqrt{2}} \ne 0$ and finite

Since $\sum v_n$ is divergent, so $\sum |u_n|$ is divergent. (by limit form comparison test)

Hence $\sum u_n$ is not absolutely convergent.

Example: Test for convergence the series $\sum_{n=1}^{\infty}(-1)^n \frac{\sin n\alpha}{n^3}$, α being real number.

Solution: We have $|u_n| = \frac{|\sin n\alpha|}{n^3} \le \frac{1}{n^3}\ \forall n$. $\left(\because \sum 1/n^3 \text{ is convergent}\right)$

Thus $\sum u_n$ is absolutely convergent for all real values of α and so $\sum u_n$ is convergent for all real values of α.

Example: Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{\sin nx + \cos nx}{n^{3/2}}$.

Solution: We have

$$|u_n| = \left|\frac{\sin nx + \cos nx}{n^{3/2}}\right| \le \frac{1}{n^{3/2}} + \frac{1}{n^{3/2}} = \frac{2}{n^{3/2}} \qquad \{\because |\sin nx| \le 1,\ |\cos nx| \le 1\ \forall x\}$$

Since $\sum \frac{1}{n^{3/2}}$ is convergent, so by First Comparison Test, $\sum |u_n|$ is convergent *i.e.,* $\sum u_n$ is absolutely convergent.

Hence $\sum u_n$ is convergent.

Example: Test for convergence and absolute convergence the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\log(n+1)} = \frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \dots$

Solution: We have $u_n = \frac{1}{\log(n+1)}\ \forall n \in \mathbb{N}$

Clearly, $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\log(n+1)} = 0$.

Since $\log x$ is an increasing function for all $x > 0$, therefore

$$\log(n+2) > \log(n+1) \qquad (\because n+2 > n+1)$$

$$\Rightarrow \quad \frac{1}{\log(n+2)} < \frac{1}{\log(n+1)}\ \forall n \Rightarrow u_{n+1} < u_n\ \forall n$$

Thus the two conditions of Leibnitz Test are satisfied and so the given series is convergent.

Now we test the absolute convergence of the given series.

We have $\sum_{n=2}^{\infty}|u_n| = \sum_{n=2}^{\infty}\frac{1}{\log n}$, which is divergent.

Hence $\sum u_n$ is not absolutely convergent i.e., $\sum u_n$ is conditionally convergent.

Example: The series $\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \dots$ converges.

Solution: We have $u_n = \frac{\log(n+1)}{(n+1)^2}\ \forall n \in \mathbb{N}$.

Since, $\lim_{n\to\infty}\frac{\log(n+1)}{(n+1)^2} = 0$.

$$\Rightarrow \lim_{n\to\infty} u_n = 0$$

Now we shall prove that $u_{n+1} \le u_n \forall n$.

Let $f(x)=\frac{\log x}{x^2}$ so that $f'(x)=\frac{x^2(1/x)-2x\log x}{x^4}$

$$\Rightarrow \ f'(x)=\frac{1-2\log x}{x^3}<0 \ \ \forall\ x>e^{1/2}\ \left(\because x>e^{1/2}\Leftrightarrow \log x>\frac{1}{2}\Leftrightarrow 1-2\log x<0\right)$$

$$\Rightarrow \qquad f(x) \text{ is decreasing function } \forall\ x>e^{1/2}$$

$$\Rightarrow \qquad f(n+2)\le f(n+1)\ \ \forall\ n \qquad\qquad \left(\because n+2>n+1>e^{1/2}\ \forall\ n\right)$$

$$\Rightarrow \qquad \frac{\log(n+2)}{(n+2)^2}\le\frac{\log(n+1)}{(n+1)^2}\ \ \forall n$$

$$\Rightarrow \qquad u_{n+1}\le u_n\ \ \forall\ n.$$

Thus both the conditions of Leibnitz test are satisfied and so the given series is convergent.

CHAPTER 5

FUNCTION AND THEIR PROPERTIES

5.1. Definitions

Function: Let $A, B \subset \mathbb{R}$ be two subsets of $\mathbb{R}$.

A function from A to B denoted by $f : A \to B$ is a rule which assign every element of A to unique element of B. A is defined as a domain of f & B is defined as co-domain of f and $f(A)$ is defined as range of f, where $f(A) = \{f(x) \in B : x \in A\}$.

1. **Equality of two functions:** Two functions f_1 & f_2 are said to be equal if and only if
   (i) f_1 & f_2 have same domain D (say)
   (ii) $f_1(x) = f_2(x) \quad \forall\, x \in D$
2. **Composition of functions:** Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that $f(x) = y$ and $g(y) = z$, where $x \in X,\ y \in Y,\ z \in Z$. Then the function $h : X \to Z$ such that $h(x) = z = g(y)$ $= g(f(x))\ \forall\, x \in X$ is known as the composition of f and g and is denoted by gof.
3. **Inverse function:** Let $f : X \to Y$ be a one-one onto function. Then the function $g : Y \to X$ which associates to each element $y \in Y$ the unique element $x \in X$ such that $f(x) = y$ is known as inverse function of f. The inverse function g of f is denoted by. f^{-1}. Then, we have $f^{-1} : Y \to X$ such that $f^{-1}(y) = x$, where $f(x) = y$.
4. **Domain of Definition:** let $y = f(x)$ be a rule, $S \subseteq \mathbb{R}$ on which f becomes real valued function i.e., if S be a subset of $\mathbb{R}$ & $f : S \to \mathbb{R}$ be real valued function then S is called domain of definition. We sometimes denote domain of definition as 'Dod'

   Examples:

   (a) $f(x) = \sin x,\ x \neq 1,\ Dod = \mathbb{R} \sim \{1\}$

   (b) $f(x) = \dfrac{1}{x},\ Dod = \mathbb{R} \sim \{0\}$

   (c) $f(x) = \log_e x,\ Dod = \mathbb{R}^+$

   (d) $f(x) = \log_e \log_e x,\ Dod = (1, \infty)$

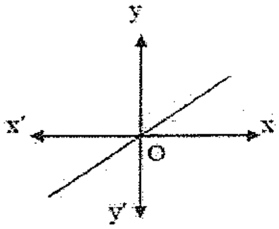
   (e) $f(x) = \log_e \log_e \log_e \log_e x,\ Dod = (e^e, \infty)$

Some well known functions

Put Your Own Notes

1. **Identity Function:** The function defined by $f(x) = x$

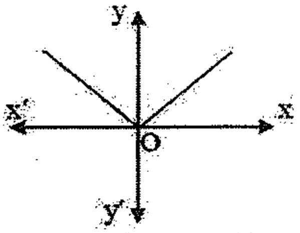
for all $x \in \mathbb{R}$ is called the identity function on $\mathbb{R}$. Clearly, the domain and range of the identity function are both equal to $\mathbb{R}$. The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° to x-axis.



2. **Polynomial Function:** A function of the form $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, where $n \in \mathbb{N}$ and $a_0, a_1, a_2, ..., a_n \in \mathbb{R}$. Then, f is called a polynomial function.

Example: $f(x) = x^3 + 2x^2 + 9$

3. **Modulus Function:** The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$ is called the modulus function. The range is the set of all non-negative real numbers.



Properties:

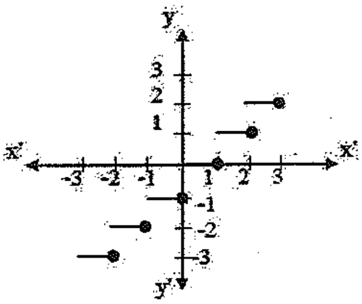
(i) $|x| \le a \Rightarrow a \le x \le a$; $(a \ge 0)$

(ii) $|x| \ge a \Rightarrow x \le -a$ or $x \ge a$; $(a \ge 0)$

(iii) $|x \pm y| \le |x| + |y|$

(iv) $|x \pm y| \ge ||x| - |y||$.

4. **The Greatest Integer Function:** Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x] = n \ \forall n \le x < n + 1, n \in \mathbb{N}$, $[x]$ indicates the integral part of x which is nearest and smallest integer to x. It is also known as floor of x. Domain of f is the set of real number and range of f is the set of integers.



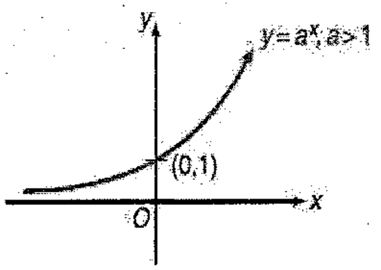
Example: $\lceil 2.3023 \rceil = 3$, $\lceil -8.0725 \rceil = -8$

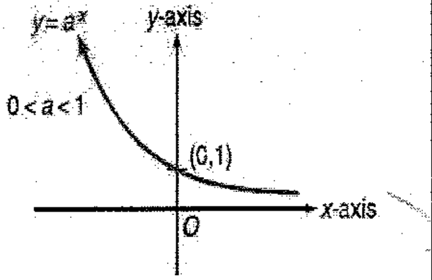
Properties:

(i) $\lceil x \rceil = x$, if $x \in \mathbb{Z}$.

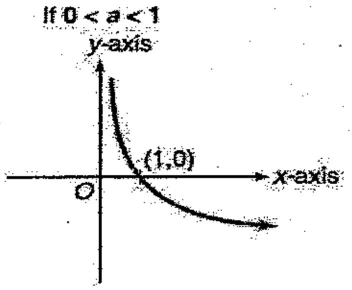
(ii) $\lceil x + I \rceil = \lceil x \rceil + I$; if $I \in \mathbb{Z}$

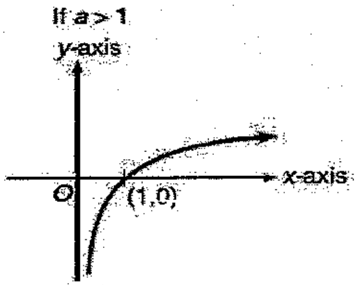
7. **Exponential Function:** Let a be positive real number. Then the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = a^x$ is called the exponential function. The range is the set of all positive real numbers.





8. **Logarithmic Function:** Let 'a' be a positive real number, then the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined as $f(x) = \log_a x$ is called the logarithmic function. The range is the set $\mathbb{R}$ of all real numbers.





Properties: Let a, b, c be positive real numbers

(i) $\log_e(ab) = \log_e a + \log_e b$

(ii) $\log_e\left(\frac{a}{b}\right) = \log_e a - \log_e b$

(iii) $\log_e a^m = m \log_e a$ $\quad m \in \mathbb{R}$

(iv) $\log_a a = 1$ $\quad a \neq 1$

(v) $\log_{b^m} a = \frac{1}{m}\log_b a$ $\quad b \neq 1$ & $m \in \mathbb{R}$

(vi) $\log_b a = \frac{1}{\log_a b}$ $\quad a, b \neq 1$

(vii) $\log_b a = \frac{\log_m a}{\log_m b}$ $\quad a, b \neq 1$ & $m > 0$

(viii) $a^{\log_a m} = m$ $\quad m > 0$ & $a \neq 1$

(ix) $a^{\log_c b} = b^{\log_c a}$ $\quad c \neq 1$

9. **Rational Function:** Let $p(x), q(x)$ be two polynomials. Then a rational function is a function $f: D \to \mathbb{R}$ defined as $f(x) = \frac{p(x)}{q(x)}$, where $D = \mathbb{R} \sim \{x \in \mathbb{R} | q(x) = 0\}$.

Example:

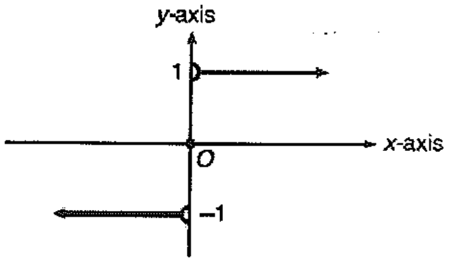
(a) Let $f: \mathbb{R} \sim \{0\} \to \mathbb{R}$ defined as $f(x) = \frac{1}{x}$

(b) Let $f: \mathbb{R} \sim \{0\} \to \mathbb{R}$ defined as $f(x) = \frac{1}{x^2}$.

10. **Signum Function:** Let $f: \mathbb{R} \to \{-1, 0, 1\}$ defined as

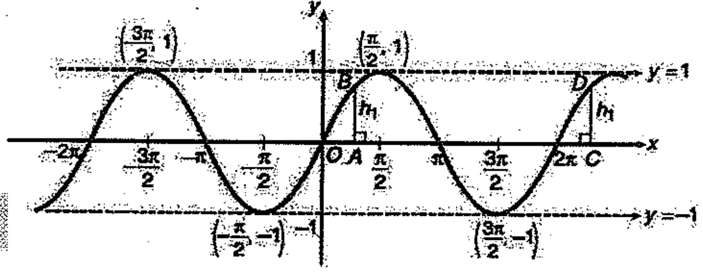
$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is called signum function.

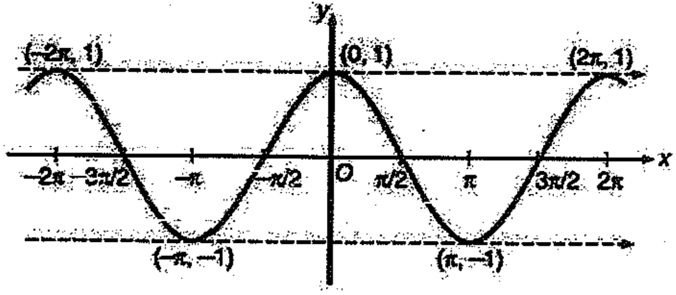


11. **Trigonometric Functions:**

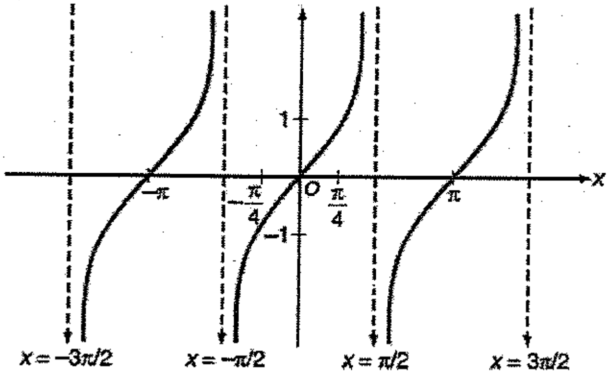
(i) **Sine Function:** The function which associates each real number x to $\sin x$ is called the sine function. Here x is the radian measure of the angle. The domain of the sine function is $\mathbb{R}$ and the range is $[-1, 1]$.



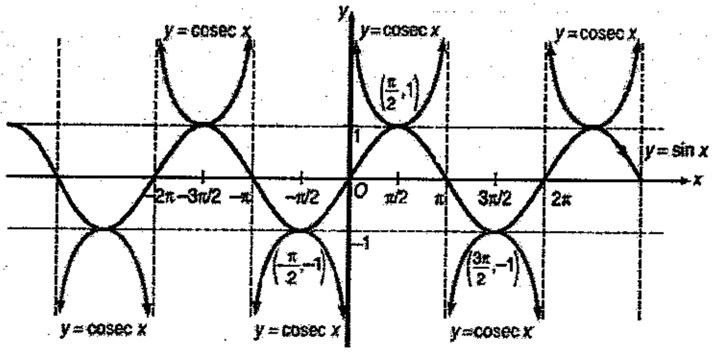
(ii) **Cosine Function:** The function that associates each real number x to $\cos x$ is called the cosine function. Here x is the radian measure of the angle. The domain of the cosine function is $\mathbb{R}$ and the range is $[-1, 1]$.



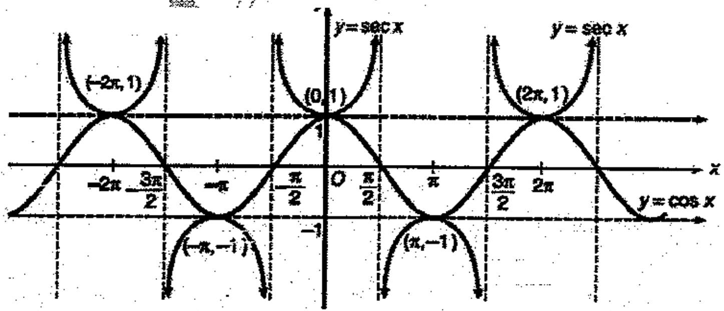
(iii) **Tangent Function:** The function that associates each real numbers to $\tan x$ is called the tangent function. Clearly, the tangent function is not defined at odd multiple of $\pi/2$, i.e. $\pm\pi/2, \pm 3\pi/2$ etc. So, the domain of the tangent function is $\mathbb{R}-\{(2n+1)\pi/2 \mid n\in\mathbb{Z}\}$. Since, it takes every value between $-\infty$ and ∞. So, the range is $\mathbb{R}$.



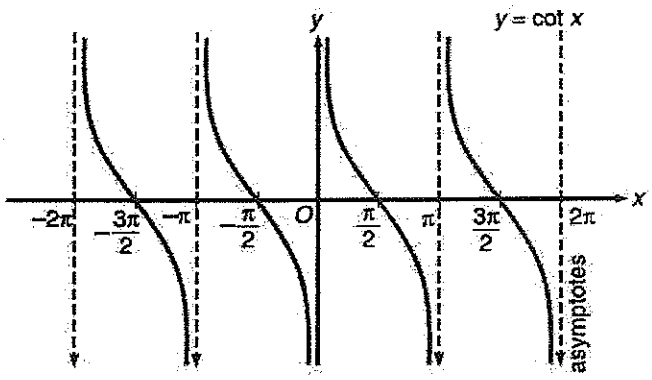
(iv) **Cosecant Function:** The function that associates each real number x to $\operatorname{cosec} x$ is called the cosecant function. Clearly, $\operatorname{cosec} x$ is not defined at $x = n\pi,\ n\in\mathbb{Z}$ i.e., $0, \pm\pi, \pm 2\pi, \pm 3\pi$ etc. So its domain is. $\mathbb{R}-\{n\pi \mid n\in\mathbb{Z}\}$. Since $\operatorname{cosec} x \ge 1$ or $\operatorname{cosec} x \le -1$. Therefore, range is $(-\infty,-1]\cup[1,\infty)$.



(v) **Secant Function:** The function that associates each real number x to $\sec x$ is called the secant function. Clearly set is not defined at odd multiples of $\pi/2$ i.e., $(2n+1)\pi/2$, $n\in\mathbb{Z}$. So, its domain is $\mathbb{R}-\{(2n+1)\pi/2 \mid n\in\mathbb{Z}\}$. Since $|\sec x|\ge 1$, therefore its range is $(-\infty,-1]\cup[1,\infty)$.



(vi) **Cotangent Function:** The function that associates a real number x to $\cot x$ is called the cotangent function. Clearly, $\cot x$ is not defined at $x = n$, $n \in \mathbb{Z}$ i.e., at $n = 0, \pm\pi, \pm 2\pi$ etc. So domain of $\cot x$ is $\mathbb{R} - \{n\pi | n \in \mathbb{Z}\}$. The range of $f(x) = \cot x$ is $\mathbb{R}$ as is evident from its graph in figure.



12. **Inverse Trigonometric Function:** The inverse trigonometric function $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$ etc. are defined as the inverse of the corresponding trigonometric functions. For example, $\sin^{-1} x$ is defined as the angle whose sine is x. But since for each value of $\sin^{-1} x$, there are unlimited number of values of the angles whose sine is x and for values of x for which $|x| > 1$, $\sin^{-1} x$ does not exist. Thus, we cannot define the inverse function of $\sin x$ unless we modify the definition of $\sin x$ in such a way that it becomes a bijection. If we consider $\sin x$ as a function from $[-\pi/2, \pi/2]$ to $[-1, 1]$, then it is a bijection. Consequently $\sin^{-1} x$ exists and it is a function from $[-1, 1]$ to $[-\pi/2, \pi/2]$.

Similarly, the other inverse trigonometric functions can be defined.

The domain and range of the inverse trigonometric functions are stated below:

Function	**Domain**	**Range**	**Definition of the function**
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$y = \sin^{-1} x \Leftrightarrow x = \sin y$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	$y = \cos^{-1} x \Leftrightarrow x = \cos y$
$\tan^{-1} x$	$(-\infty, \infty)$ or $\mathbb{R}$	$(-\pi/2, \pi/2)$	$y = \tan^{-1} x \Leftrightarrow x = \tan y$
$\cot^{-1} x$	$(-\infty, \infty)$ or $\mathbb{R}$	$(0, \pi)$	$y = \cot^{-1} x \Leftrightarrow x = \cot y$
$\text{cosec}^{-1} x$	$\mathbb{R} - (-1, 1)$	$[-\pi/2, \pi/2] - \{0\}$	$y = \text{cosec}^{-1} x \Leftrightarrow x = \text{cosec}\, y$
$\sec^{-1} x$	$\mathbb{R} - (-1, 1)$	$[0, \pi] - \{\pi/2\}$	$y = \sec^{-1} x \Leftrightarrow x = \sec y$

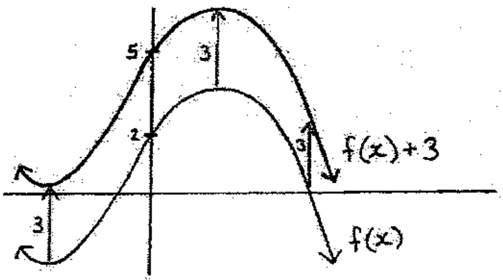
5.2. Graphical Transformation

1. **When $f(x)$, transforms to $f(x)\pm a$. (where a is +ve)**

   i.e., $f(x)\rightarrow f(x)+a$

   shift the given graph of $f(x)$ upward through 'a' units again, $f(x)\rightarrow f(x)-a$ shift the given graph of $f(x)$ downward through 'a' units

   Example:

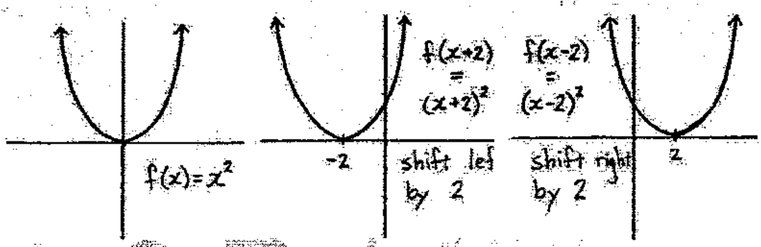


2. **$f(x)$ transforms to $f(x\pm a)$**

   i.e., $f(x)\rightarrow f(x-a)$; a is positive. Shift the graph of $f(x)$ through 'a' unit towards right $f(x)$ transformations to $f(x+a)$.

   i.e., $f(x)\rightarrow f(x+a)$; a is positive. Shift the graph of $f(x)$ through 'a' unit towards left.

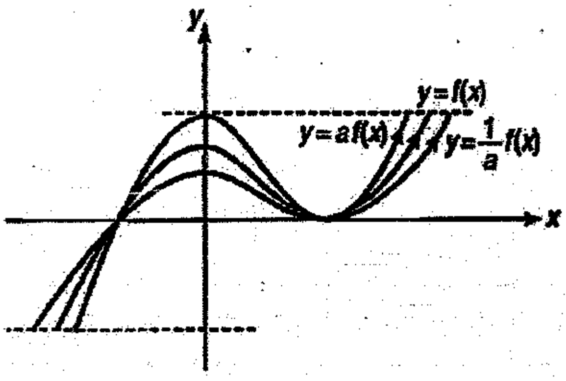
   Example:



3. **$f(x)$ transforms to $af(x)$**

   i.e., $f(x)\rightarrow af(x)$; $a>1$ Stretch the graph of $f(x)$ 'a' times along y-axis.

   $f(x)\rightarrow\frac{1}{a}f(x)$; $a>1$. Shrink the graph of $f(x)$ 'a' times along y-axis.



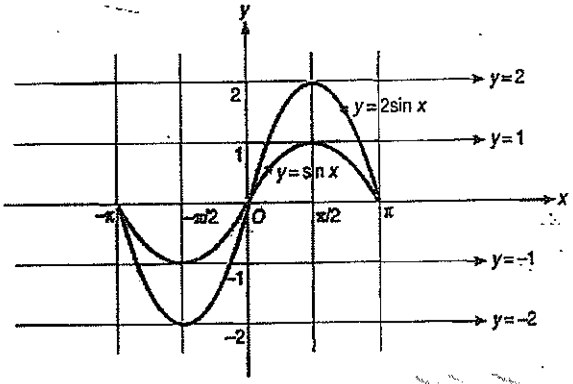
Put Your Own Notes

Example: Plot $y = \sin x$ and $y = 2\sin x$.

Solution: We know; $y = \sin x$ and $f(x) \to a\,f(x)$

$\Rightarrow$ Stretch the graph of $f(x)$ 'a' times along y-axis.

$\therefore$ $y = 2\sin x$ $\Rightarrow$ stretch the graph of $\sin x$ '2' times along y-axis.



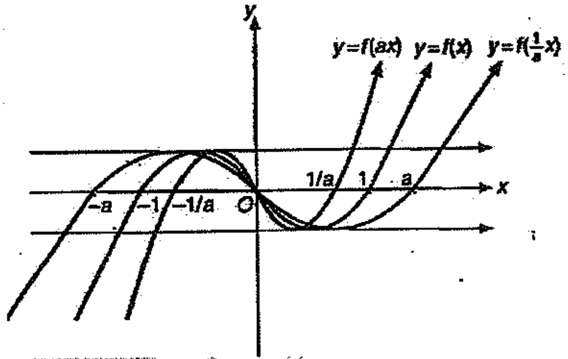
4. $f(x)$ transforms to $f(ax)$

i.e., $f(x) \to f(ax)$; $a > 1$

Stretch (or contract) the graph of $f(x)$ 'a' times along y-axis.

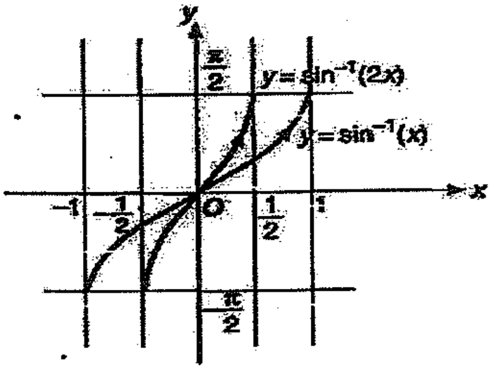
again $f(x) \to f\left(\dfrac{1}{a}x\right)$; $a > 1$.

Shrink (or expand) the graph of $f(x)$ 'a' times along y-axis.



Example: Plot $y = \sin^{-1} x$ and $y = \sin^{-1}(2x)$.

Solution: Here; $y = \sin^{-1}(2x)$, is to shrink (or contract) the graph of $\sin^{-1} x$ '2' times along x-axis.



5. **$f(x)$ transforms to $f(-x)$**

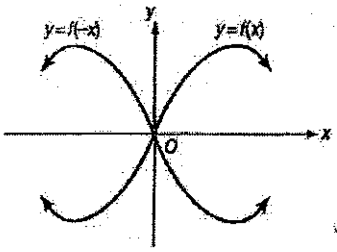
i.e., $f(x) \to f(-x)$

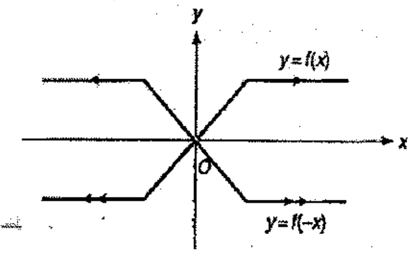
To draw $y = f(-x)$, take the image of the curve $y = f(x)$ in y-axis as plane mirror.

OR

"Turn the graph of $f(x)$ by 180° about y-axis."

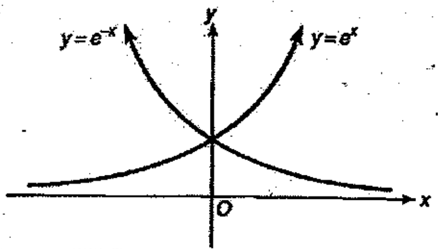
OR





Example: Plot $y = e^{-x}$.

Solution: As $y = e^x$ is known; they $y = e^{-x}$ is the image in y-axis as plane mirror for $y = e^x$; shown as;



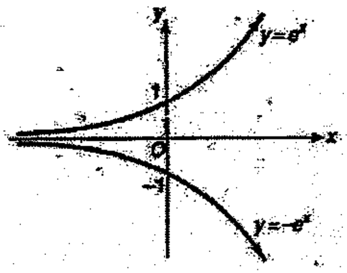
6. **$f(x)$ transforms to $-f(x)$**

i.e., $f(x) \to -f(x)$;

To draw $y = -f(x)$, take image of $y = f(x)$ in the y- axis as plane mirror.

OR

"Turn the graph of $f(x)$ by 180° about x- axis."



Example: Plot the curve $y=-e^x$.

Solution: As $y=e^x$ is known;

$\therefore$ $y=-e^x$ take image of $y=e^x$ in the x-axis as plane mirror.

7. **$f(x)$ transforms to $-f(-x)$**

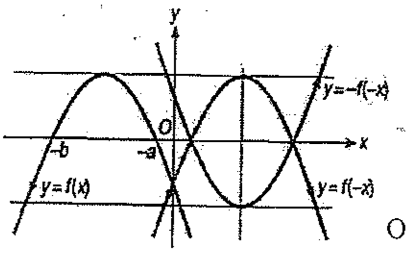
i.e., $f(x)\rightarrow -f(-x)$;

To draw $y=-f(-x)$ take image of $f(x)$ about y-axis to obtain $f(-x)$ and then take image of $f(-x)$ about x-axis to obtain $-f(-x)$.

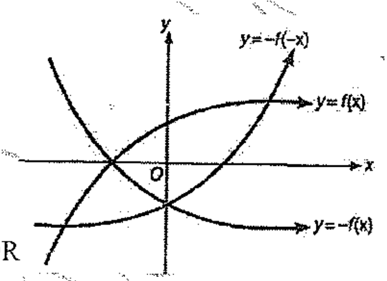
$\therefore f(x)\rightarrow -f(-x)$

(i) Image about y-axis

(ii) Image about x-axis



OR

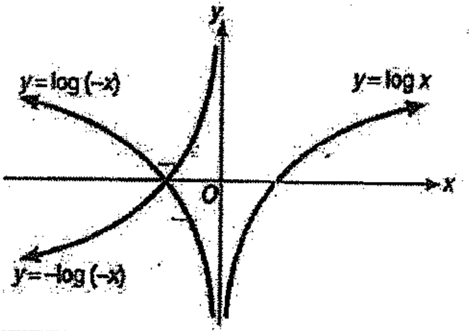


Example: Plot the curve $y=-\log(-x)$.

Solution: As $y=\log x$ is known;

(i) Take image about y-axis, for $y=\log(-x)$.

(ii) Take image of $y=\log(-x)$ about x-axis, for $y=-\log(-x)$.



8. **$f(x)$ transforms to $|f(x)|$**

i.e., $f(x)\rightarrow|f(x)|$

Here; $y=|f(x)|$ is drawn in two steps.

(i) In the first step, leave the positive part of $f(x)$, {*i.e.*, the part of $f(x)$ above x-axis) as it is.

(ii) In the second step, take the mirror image of negative part of $f(x)$. {*i.e.*, the part of $f(x)$ below x-axis) in the x-axis as plane mirror.

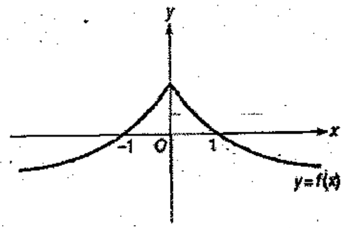
OR

Take the mirror image (in x-axis) of the portion of the graph of $f(x)$ which lies below x-axis.

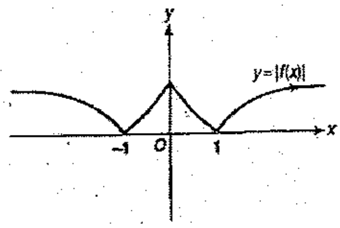
OR

Turn the portion of the graph of $f(x)$ lying below x-axis by 180° about x-axis.

Graph of $f(x)$:



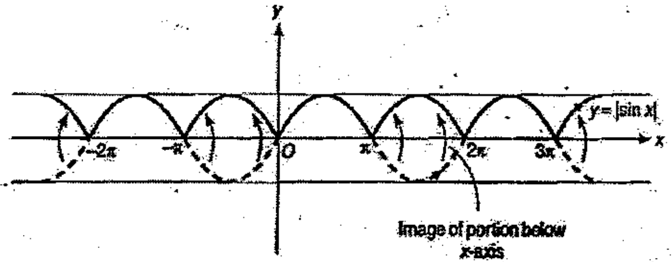
Graph for $|f(x)|$:



Example: Sketch the graph for $y = |\sin x|$.

Solution: Here; $y = \sin x$ is known.

$\therefore$ To draw $y = |\sin x|$, we take the mirror image (in x-axis) of the portion of the graph of $\sin x$ which lies below x-axis.



9. $f(x)$ transforms to $f(|x|)$

i.e., $f(x) \to f(|x|)$

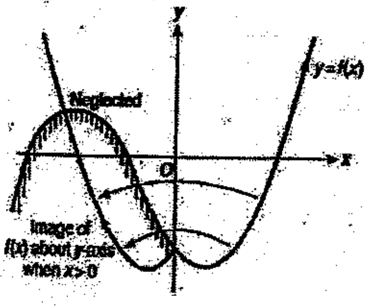
If we know $y = f(x)$, then to plot $y = f(|x|)$, we should follow two steps:

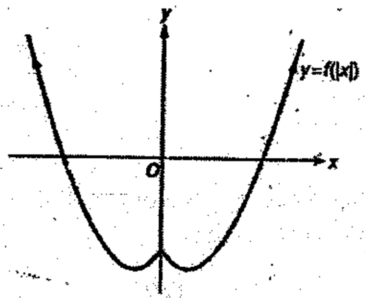
(i) Leave the graph lying right side of the y-axis as it is.

(ii) Take the image of $f(x)$ in the right of y-axis as the plane mirror and the graph of $f(x)$ lying leftward of the y-axis (if it exists) is omitted.

OR

Neglect the curve for $x < 0$ and take the images of curve for $x \geq 0$ about y-axis.





Put Your Own Notes

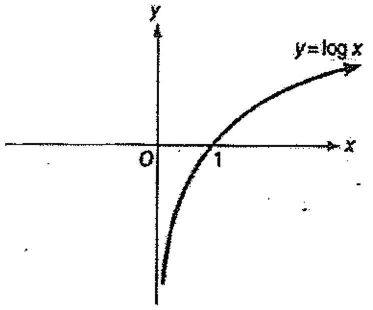
Example: Sketch the graph for $y = \log|x|$.

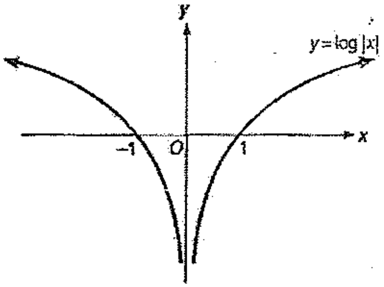
Solution: As we know, the curve $y = \log x$.

$\therefore$ $y = \log|x|$ could be drawn in two steps:

(i) Leave the graph lying right side of the y-axis as it is.

(ii) Take the image of $f(x)$ in the right of y-axis as the plane mirror.





10. Sketching $h(x) = $ Minimum $\{f(x), g(x)\}$ and

$h(x) = $ Minimum $\{f(x), g(x)\}$

(i) $h(x) = \text{Maximum } \{f(x), g(x)\}$

$$\Rightarrow h(x) = \begin{cases} f(x); & \text{when } f(x) > g(x) \\ g(x); & \text{when } g(x) > f(x) \end{cases}$$

$\therefore$ Sketch $f(x)$ when its graph is above the graph of $g(x)$ and sketch $g(x)$ when its graph is above the of $f(x)$.

(ii) $h(x) = \text{Minimum } \{f(x), g(x)\}$

$$\Rightarrow h(x) = \begin{cases} f(x), & \text{when } f(x) < g(x) \\ g(x), & \text{when } g(x) < f(x) \end{cases}$$

$\therefore$ Sketch $f(x)$ when its graph is lower and otherwise sketch $g(x)$.

OR

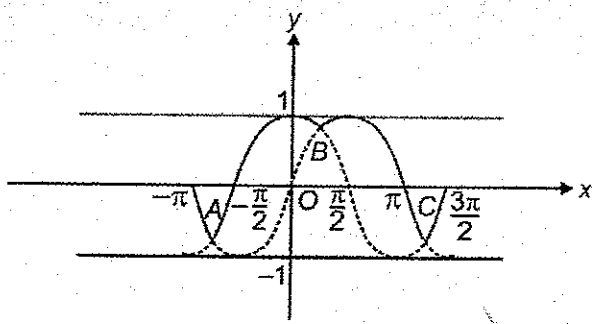
"To draw the graph of functions of the form $y = \max\{f(x), g(x)\}$ or $y = \min\{f(x), g(x)\}$."

We first draw the graphs of both the functions $f(x)$ and $g(x)$ and their points of intersections. Then we find any two consecutive points of intersection. In between these points either $f(x) > g(x)$ or $f(x) < g(x)$, then, in order to $\max\{f(x), g(x)\}$ we take those segments of $f(x)$ for which $f(x) > g(x)$, between any two consecutive points of intersection of $f(x)$ and $g(x)$.

Similarly, in order to $\min\{f(x), g(x)\}$, we take those segments of $f(x)$ for which $f(x) < g(x)$, between any two consecutive points of intersection of $f(x)$ and $g(x)$.

Example: Sketch the graph of $y=\max\{\sin x,\cos x\},\ \forall x\in\left(-\pi,\frac{3\pi}{2}\right)$.

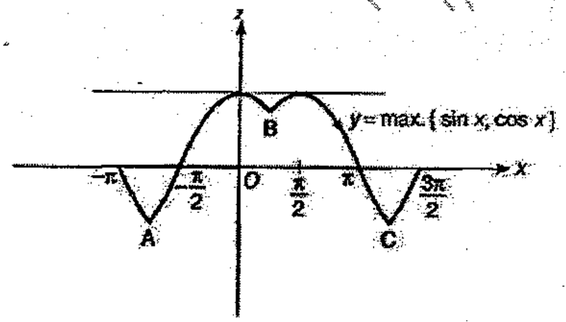
Solution:



First plot both $y=\sin x$ and $y=\cos x$ by a dotted curve as can be seen from the graph in the interval $\left(-\pi,\frac{3\pi}{2}\right)$ and then darken those dotted lines for which $f(x)>g(x)$ or $g(x)>f(x)$.

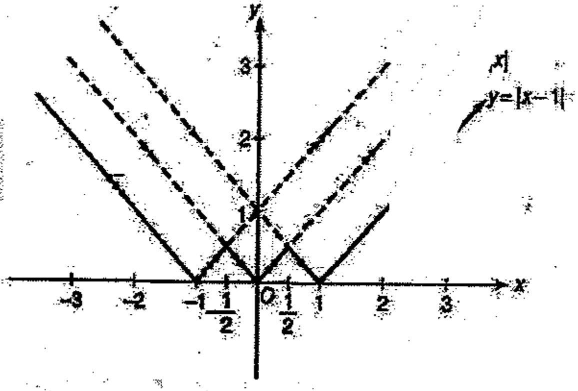
From adjacent figure the point of intersections are A,B,C.

$\therefore$ **Graph of $\max\{\sin x,\cos x\}$**



Example: Sketch the graph for $y=\min\{|x|,|x-1|,|x+1|\}$.

Solution:



First plot the graph for:

$y=|x|, y=|x-1|, y=|x+1|$ by a dotted curve as can be seen from the graph and then darken those dotted lines for which

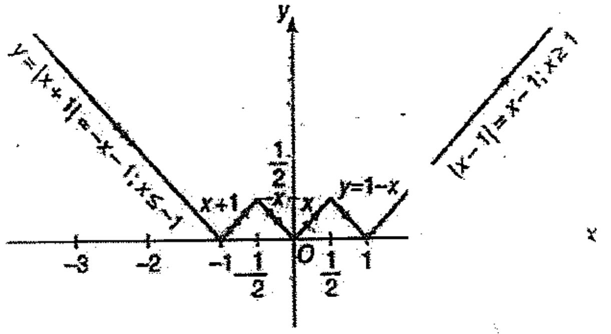
$|x|<\{|x-1|,|x+1|\}$:

$|x-1|<\{|x|,|x+1|\}$

and $|x+1|<\{|x|,|x-1|\}$.

Graph for $y=|x|, y=|x-1|, y=|x+1|$.

As from the above curve graph for $y = \min\{|x-1|, |x|, |x+1|\}$ is plotted as;

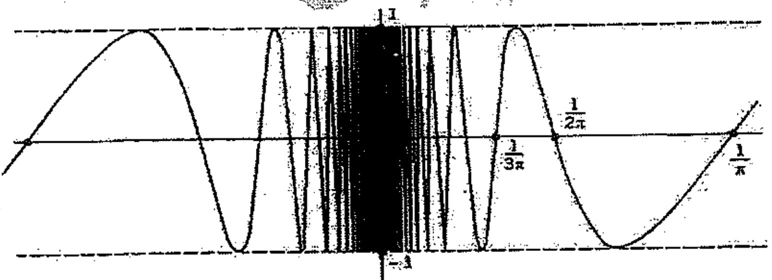


From above figure : $\min\{|x-1|, |x|, |x+1|\} = \begin{cases} -(x+1) & ; \ x \le -1 \\ (x+1) & ; \ -1 \le x \le -\frac{1}{2} \\ -(x) & ; \ -\frac{1}{2} \le x \le 0 \\ (x) & ; \ 0 \le x \le \frac{1}{2} \\ (1-x) & ; \ \frac{1}{2} \le x \le 1 \\ (x-1) & ; \ x \ge 1 \end{cases}$

5.3. Some Important Functions and Their Graphs

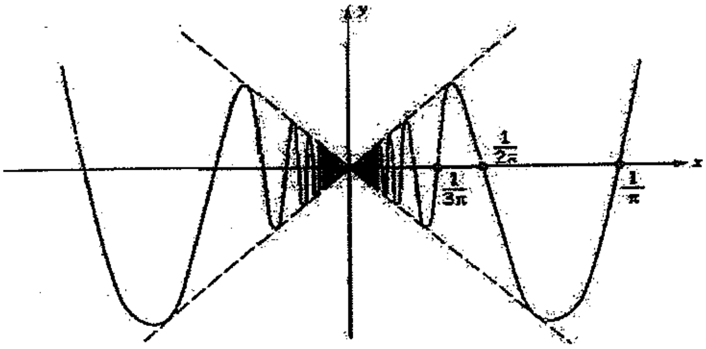
1. $f : A \to \mathbb{R}$

$f(x) = \sin\left(\frac{1}{x}\right) \quad \forall x \in A$



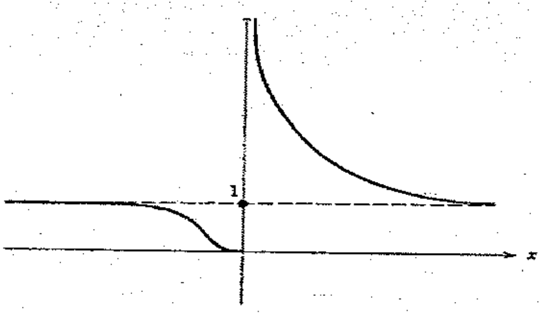
2. $f : A \to \mathbb{R}$

$f(x) = x\sin\left(\frac{1}{x}\right) \quad \forall x \in A$



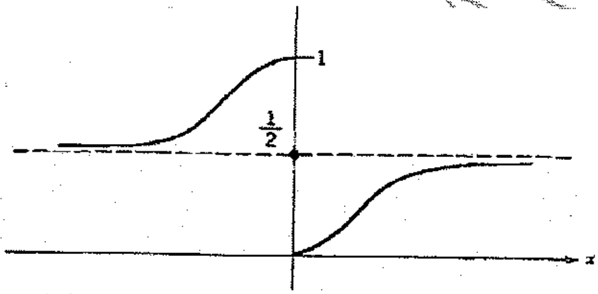
3. $f : A \to \mathbb{R}$

$f(x) = e^{1/x} \qquad \forall x \in A$



4. $f : A \to \mathbb{R}$

$f(x) = \left(\frac{1}{e^{1/x}+1}\right) \forall x \in A$



Put Your Own Notes

5.4. Classification of Functions

1. **Algebraic Function:** A function $y = f(x)$ is said to be an algebraic function and set $S \subset \mathbb{R}$, if it is a root of the equation of the form

   $p_0(x)y^n + p_1(x)y^{n-1} + \ldots + p_n(x) = 0$

   Where $p_i(x)$ are polynomial in $x \in S \quad o \le i \le n$

   Example:

   (a) $p(y) = -3xy^2 - (1+x^2)y + 2x = 0$

   Then $y = \dfrac{(1+x^2) \pm \sqrt{(1+x^2)^2 + 4(3x)(2x)}}{-6x}$

   (b) Constant functions are algebraic function even if $f(x) = \pi$.

2. **Transcendental Function:** If a function $y = f(x)$ defined on a set $S \subset \mathbb{R}$ is not an algebraic function then it is said to be a transcendental function on S.

   Examples:

   (a) $f(x) = x^{\pi}$

   (b) $f(x) = c^x,\ c \ne 0, 1$

   (c) $f(x) = x^x$

   (d) $f(x) = x^{\frac{1}{x}}$

   (e) $f(x) = \log_c x,\ c \ne 0, 1$

   (f) $f(x) = \sin x$

3. **Periodic Function:** A function $y = f(x)$ is said to be periodic if $\exists\ l \in \mathbb{R}$ such that $f(x+l) = f(x) \quad \forall\, x \in D.o.d$ and $x + l \in D.o.d.$ and such l is called a period of f.

Fundamental Period: The smallest positive $\alpha \in \mathbb{R}$ for a periodic function f is defined as the fundamental period of f.

Examples:

(a) $f(x) = \sin x$ is a periodic function with set of periods $\{2n\pi : n \in \mathbb{Z}\}$

fundamental period of $f(x) = \sin x$ is 2π

(b) $f(x) = \tan x$ is a periodic function with fundamental period π.

Remark:

(a) Periodicity is a domain based property.

(b) A periodic function may or may not have fundamental period.

Example: $f : \mathbb{R} \to \mathbb{R}$

$f(x) = c\ \forall x \in \mathbb{R},\quad c \in \mathbb{R}$

Then for any $\alpha \in \mathbb{R}$, $f(x+\alpha) = f(x) = c \qquad \forall\, x \in \mathbb{R}$

set of periods of f is $\mathbb{R}$

but fundamental period does not exist.

(c) If a function f has a fundamental period T. Then $af(bx+c)+d$ has period $\dfrac{T}{|b|}$, $a, b, c, d \in \mathbb{R}$.

(d) There exist a periodic function whose fundamental period does not exist but set of period is dense in $\mathbb{R}$

Example: $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{Q}^c \end{cases}$$

Then $f(x+\alpha) = f(x) \quad \forall\, x \in \mathbb{R},\ \forall\, \alpha \in \mathbb{Q}$.

$\therefore$ Set of periods is $\mathbb{Q}$ (set of rationals)

But fundamental period does not exist.

(e) If $f_1, f_2, \ldots, f_n$ are periodic functions with periods $T_1, T_2, \ldots, T_n$ respectively and L.C.M. of $T_1, T_2, \ldots, T_n$ exist. Then their linear combination of $f_i's$ is periodic function and their L.C.M. decide the period of $f_i's$.

(f) $$L.C.M\left\{\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n}\right\} p_i, q_i \in \mathbb{N} = \frac{L.C.M\{p_1, p_2, \ldots, p_n\}}{H.C.F\{q_1, q_2, \ldots, q_n\}}$$

(g) If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are irrational numbers, obtained by multiplying same irrational number by different rational (non zero) then

$L.C.M(\alpha_1, \alpha_2, \ldots, \alpha_n) = L.C.M$ (Rational multiplied by that irrational)

(h) L.C.M of rational and irrational numbers is not defined.

Examples of Periodic Functions:

(a) $f(x)=\sin\left(\frac{x}{3}\right)$ is a periodic function with period 6π.

Reason: Since period of $\sin x = 2\pi$

$\therefore$ period of $\sin\left(\frac{x}{3}\right)=2\pi\times 3=6\pi$

(b) $f(x)=\sin^2 x$ has period π.

Reason: Since $\sin^2 x=\frac{1-\cos 2x}{2}=\frac{1}{2}-\frac{1}{2}\cos(2x)$

as $\cos 2x$ has period π

$\therefore \sin^2(x)$ has period $\frac{2\pi}{2}=\pi$

(c) $f(x)=\sin^3 x$ has a period 2π

Reason: $\sin^3 x=\frac{1}{4}(3\sin x-\sin 3x)$

$=\frac{3}{4}\sin x-\frac{1}{4}\sin 3x$

since $\frac{3}{4}\sin x$ has period 2π

$\frac{1}{4}\sin 3x$ has period $\frac{2\pi}{3}$

and $L.C.M\left\{2\pi,\frac{2\pi}{3}\right\}=L.C.M\left\{2,\frac{2}{3}\right\}$

$=\frac{L.C.M\{2,2\}}{H.C.F\{1,3\}}$

$=2$

$\therefore \sin^3 x$ has period 2π.

(d) $f(x)=\sin\pi x+x-[x]$ is periodic function with period 2.

Reason: Since $\sin(\pi x)$ has period $\frac{2\pi}{\pi}=2$

and $x-[x]$ has period 1.

also $L.C.M\{2,1\}$ is 2.

$\therefore f(x)$ is periodic function with period 2.

Notes:

(a) Sum or difference of two periodic function may not be periodic

Example: Take $f(x)=\sin x,\ g(x)=x-[x]$

Then f and g are periodic function with period 2π and 1 resp.

But $f(x)\pm g(x)$ is not periodic.

(b) Sum of two non periodic function may be periodic function.

Example: $f(x)=x^2,\ g(x)=-x^2 \quad \forall\, x\in\mathbb{R}$

Then f and g both are non periodic functions

But $(f+g)(x)=f(x)+g(x)=0 \quad \forall\, x\in\mathbb{R}$

is a periodic function.

(c) Sum of two periodic functions (having their fundamental periods) may be periodic but fundamental period may not exist

Example: $f(x)=\sin^2 x, g(x)=\cos^2 x$ both are periodic functions with fundamental period π. But $f(x)+g(x)=\sin^2 x+\cos^2 x=1$ does not have fundamental period.

(d) If f and g are two functions defined on $\mathbb{R}$ such that g is periodic. Then fog is periodic on $\mathbb{R}$.

Reason: Since g is periodic, so $\exists\ \alpha\in\mathbb{R}$ such that $g(x+\alpha)=g(x)\ \ \forall\, x\in D \quad x+\alpha\in D$.

So $(fog)(x+\alpha)=f(g(x+\alpha))$

$=f(g(x))$

$=(fog)(x) \qquad \forall\, x\in D.o.d,\ \ x+\alpha\in D.o.d$

(e) Algebraic functions cannot be periodic unless they are constant function.

4. **Bounded Function:** Let $S\subseteq\mathbb{R}$ and $f:S\to\mathbb{R}$ be a function. Then f is bounded if and only if $f(S)$ is bounded in $\mathbb{R}$

i.e., $\exists\ \alpha,\beta\in\mathbb{R}$ such that

$\alpha\le f(x)\le\beta \qquad \forall\, x\in S$

Notes:

(a) If S is finite $\Rightarrow$ Range is finite $\Rightarrow f$ is bounded.

(b) For a function to be unbounded, range must be infinite set.

(c) f is unbounded on $S \Leftrightarrow \forall\, n\in\mathbb{N}\ \ \exists\ \ x_n\in S$ such that

$|f(x_n)|>n$.

i.e., $\langle f(x_n)\rangle$ is unbounded.

(d) Sum, difference and composition of two bounded functions is bounded function.

Example: $f:[0,1]\to\mathbb{R}$

$$f(x)=\begin{cases}\dfrac{1}{x} & x\in(0,1]\\ 0 & x=0\end{cases}$$

Then f is unbounded.

Reason: For all $n\in\mathbb{N}$, $\exists\ \dfrac{1}{n}\in[0,1]$ such that

$$f\left(\frac{1}{n}\right)=n\in f([0,1])$$

$\Rightarrow \mathbb{N}\subseteq f([0,1])$

$\Rightarrow$ Unbounded set contained in $f([0,1])$

$\Rightarrow f([0,1])$ is unbounded.

5. **Monotonic Function:** Let $S\subseteq\mathbb{R}$ and $f:S\to\mathbb{R}$ then

   (i) f is said to be monotonically increasing if for $x_1<x_2$

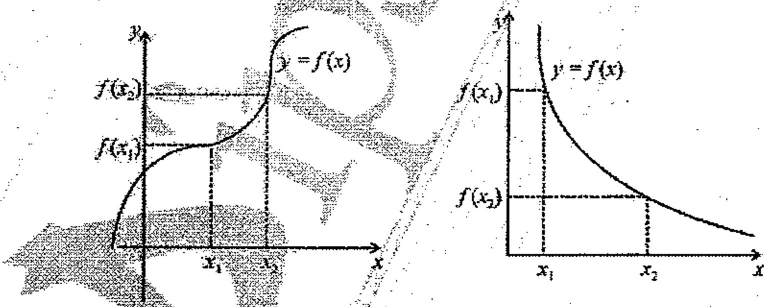
   $\Rightarrow f(x_1)\le f(x_2)\qquad \forall\ x_1,x_2\in S$

   (If strict inequality hold then f is called strictly increasing function)

   (ii) f is said to be monotonically decreasing if for $x_1<x_2$

   $\Rightarrow f(x_2)\ge f(x_1)\ \ \forall\ x_1,x_2\in S$

   If strict inequality hold then f is called strictly decreasing function.



Monotonic Functions

Notes:

(a) If f is monotonically increasing (decreasing) function. Then $-f$ is monotonically decreasing (increasing) function.

(b) If $f_1:X_1\to X_2$ is monotonic on X_1 and $f_2:X_2\to X_3$ is monotonic on X_2 then $f_2of_1:X_1\to X_3$ is monotonic on X_1.

(c) Sum of two monotonically increasing (decreasing) function is monotonically increasing (decreasing).

(d) Difference of two monotonically increasing (decreasing) function may not be monotonically increasing (decreasing).

Example: $f:\left[0,\frac{\pi}{2}\right]\to\mathbb{R}$ $g:\left[0,\frac{\pi}{2}\right]\to\mathbb{R}$

$f(x)=\sin x,\quad g(x)=x\quad\forall\, x\in\left[0,\frac{\pi}{2}\right]$

Then f and g are monotonically increasing functions but $h(x)=\sin x-x$ is not monotonically increasing function.

(e) Product of two monotonically increasing (decreasing) functions may not be monotonically increasing (decreasing).

Example: $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}0 & x<0\\ 3 & x\ge 0\end{cases}$$

and $g:\mathbb{R}\to\mathbb{R}$

$$g(x)=\begin{cases}-7 & x<0\\ -2 & x\ge 0\end{cases}$$

But $f(x).g(x)=\begin{cases}0 & x\le 0\\ -6 & x>0\end{cases}$

is not monotonically increasing.

Examples of some well known monotonic functions:

(a) $f:\mathbb{R}\to\mathbb{R}$ defined by

$f(x)=a^x\ \forall\, x\in\mathbb{R},\ a>1$ is monotonically increasing function.

(b) $f:(0,\infty)\to\mathbb{R}$ defined by

$f(x)=\log_e x\ \forall\, x\in(0,\infty)$ is monotonically increasing function.

(c) $f:\mathbb{R}\to\mathbb{R}$ defined by

$f(x)=-x\ \forall\, x\in\mathbb{R}$ is monotonically decreasing function.

6. **Even Function and Odd Function:** A function $f:S\to\mathbb{R}$ is said to be an even function if $f(-x)=f(x)\forall\, x\in S$ and f is called an odd function if $f(-x)=-f(x)\forall\, x\in S$.

Examples of even functions:

(a) $f(x)=x^2$ on $\mathbb{R}$

(b) $f(x)=\cos x$ on $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

(c) $f(x)=|x|$ on $\mathbb{R}$ are even functions.

Examples of odd functions:

(a) $f(x)=\sin x$ on $\mathbb{R}$

(b) $f(x)=x^3$ on $\mathbb{R}$

(c) $f(x)=x^2\sin(x)$ on $\mathbb{R}$ all are odd functions.

Put Your Own Notes

Put Your Own Notes

Remark:

(a) The graph of an even function is always symmetrical about y-axis. *i.e.,* the curve left hand side of y-axis is the mirror image of the curve on its right hand side.

(b) The graph of an odd function is always symmetric in opposite quadrant. *i.e.,* the curve in the first quadrant is identical to the curve in third quadrant and the curve in second quadrant is identical to the curve in fourth quadrant.

(c) If f is a even and differentiable function on $\mathbb{R}$. Then $f'(x)$ is an odd function.

(d) If $f(x)$ is an odd differentiable function on $\mathbb{R}$, then $f'(x)$ is an even function.

5.5. Limit of a Function

5.5.1 General Principle For Existence of Limit (GPEL)

Let $S \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$ and $x = \alpha$ is a limit point of S (may or may not be member of S)

Let $f : S \to \mathbb{R}$

We say $l \in \mathbb{R}$ is the limit of $f \Leftrightarrow$ for any $\varepsilon > 0 \ \exists \, \delta > 0$

Such that

$$x_1, x_2 \in \{x : 0 < |x - \alpha| < \delta\} \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$$

Note: This method is rather useful to disclaim that limit does not exist.

Examples:

1. $\lim\limits_{x \to a} \sin \dfrac{1}{x - a}$ does not exist.

Solution: For any $\delta > 0 \ \exists \, n \in \mathbb{N}$ such that

$$-\delta < x_1 - a = \frac{1}{-2n\pi - \frac{\pi}{2}} < x_2 - a = \frac{1}{2n\pi + \frac{\pi}{2}} < \delta, \text{ and so,}$$

$$x_1, x_2 \in \{x : 0 < |x - a| < \delta\} \Rightarrow \left|\sin \frac{1}{x_1 - a} - \sin \frac{1}{x_2 - a}\right| = |-1 - 1| = 2 \nless \varepsilon = 1.$$

Hence, by the general principle for the existence of limits, $\lim\limits_{x \to a} \sin \dfrac{1}{x - a}$ does not exist.

2. $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{Q}^c \end{cases}$$

Then limit does not exist at any real number.

Solution: For any $\alpha \in \mathbb{R}$ and for any $S > 0$, $\exists \, x_1 \in \mathbb{Q}$ and $x_2 \in \mathbb{Q}^c$ and $x_1, x_2 \in (\alpha - \delta, \alpha + \delta)$ such that

$$[f(x_1) - f(x_2)| = |1 + 1| = 2$$

3. Prove that $\lim\limits_{x\to a}\sqrt{x}=\sqrt{a}$ for $a>0$.

Solution: The function $f(x)=\sqrt{x}$ is defined for all non-negative real numbers. Since we consider the interval $(a-\delta, a+\delta)$, we can choose $0<\delta<a$. Now $\sqrt{x}+\sqrt{a}>\sqrt{a}$.

Now $\left|\sqrt{x}-\sqrt{a}\right|=\left|\dfrac{x-a}{\sqrt{x}+\sqrt{a}}\right|<\dfrac{1}{\sqrt{a}}|x-a|<\varepsilon$, if $|(x-a)|<\varepsilon\sqrt{a}$.

Hence, we choose $\delta=\min\left(a,\varepsilon\sqrt{a}\right)$ or any other smaller positive real number. Hence if $0<|x-a|<\delta$, then we have

$$\left|\sqrt{x}-\sqrt{a}\right|<\frac{1}{\sqrt{a}}|x-a|<\frac{\delta}{\sqrt{a}}<\frac{\varepsilon\sqrt{a}}{\sqrt{a}}=\varepsilon.$$

4. $\lim\limits_{x\to 0}x\sin\dfrac{1}{x}=0$

Solution: Here $f(x)=x\sin(1/x)$.

Let $\varepsilon>0$ be given. Then we want to find a $\delta>0$ such that

$|f(x)-0|<\varepsilon$ whenever $0<|x-0|<\delta$.

Now, $|f(x)-0|=\left|x\sin\dfrac{1}{x}\right|=|x|\left|\sin\dfrac{1}{x}\right|\le x$, as $\left|\sin\dfrac{1}{x}\right|\le 1$

We can choose $\delta=\varepsilon$. Then we have

$|f(x)-0|<\varepsilon$ whenever $0<|x|<\varepsilon$

Hence, $\lim\limits_{x\to 0}x\sin\dfrac{1}{x}=0$

5.5.2 Second Definition

Let $S\subseteq\mathbb{R}$ be a subset of $\mathbb{R}$ and $x=\alpha$ be a limit point of S (may or may not belongs to S)

Let $f:S\to\mathbb{R}$.

We say $l\in\mathbb{R}$ is the limit of $f\Leftrightarrow$ for any sequence $\langle a_n\rangle$, such that $\langle a_n\rangle\to\alpha\Rightarrow f(a_n)\to l$. $(a_n\in S\ \forall n\in\mathbb{N})$

Note:

(a) If we can find two sequences $\langle a_n\rangle$ and $\langle b_n\rangle$ in S converging to α but $f(a_n)$ and $f(b_n)$ converging to different limit. Then we say the function has no limit at $x=\alpha$ or limit does not exist at $x=\alpha$.

(b) We can say limit does not exist at $x=\alpha$ if we can find a sequence $\langle a_n\rangle$ in S such that $\langle a_n\rangle\to\alpha$ and $f(a_n)$ does not converge at all.

Examples:

(a) $f:\mathbb{R}\sim\{0\}\to\mathbb{R}$ defined by

$$f(x)=\sin\left(\frac{1}{x}\right)\forall\in\mathbb{R}\sim\{0\}$$

Then limit of f does not exist at $x=0$.

Reason: $\langle a_n\rangle=\left\langle\frac{1}{n}\right\rangle$ is a sequence in $S=\mathbb{R}\sim\{0\}$ such that

$$\langle a_n\rangle\to 0 \text{ but } f(a_n)=f\left(\frac{1}{n}\right)=\sin\left(\frac{1}{\frac{1}{n}}\right)=\sin(n)$$

does not converge anywhere.

(b) $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$$

Then limit of f does not exist at $x=0$.

Reason: Take $\langle a_n\rangle=\left\langle\frac{1}{n}\right\rangle$ and $\langle b_n\rangle=\left\langle-\frac{1}{n}\right\rangle$

Then $a_n\to 0$ and $b_n\to 0$

But $f(a_n)\to 1$ and $f(b_n)\to -1$

(c) $f:\mathbb{R}\sim\{0\}\to\mathbb{R}$ defined by

$$f(x)=\frac{1}{x}\ \forall\, x\in\mathbb{R}\sim\{0\}$$

Reason: Take $\langle a_n\rangle=\left\langle\frac{1}{n}\right\rangle$, then $\lim\limits_{x\to 0} f(x)$ does not exist.

Then $\langle a_n\rangle\to 0$ but $f(a_n)=\left\langle\frac{1}{\frac{1}{n}}\right\rangle=\langle n\rangle$ which is not convergent.

5.5.3 One Side Limit

Definition 1: A function $f(x)$ is said to approach L as $x\to a$ from the right or from above if given $\varepsilon>0$, there exists a $\delta>0$ such that

$|f(x)-L|<\varepsilon$ whenever $a<x<a+\delta$.

The number L is called the right hand limit of f at a. We write this as $\lim\limits_{x\to a+} f(x)=L$. Note that for the existence of the right hand limit $f(x)$ should be defined in $(a,a+\delta)$.

Definition 2: A function $f(x)$ is said to approach M as x approaches a from the left or from below if given $\varepsilon>0$, there exists $\delta>0$ such that

$|f(x)-M|<\varepsilon$ whenever $(a-\delta<x<a)$.

The number M is called the left hand limit of f at $x = a$. We write this as $\lim_{x \to a-} f(x) = M$. Note that for the existence of the left hand limit $f(x)$ should be defined in $(a - \delta, a)$.

Note: We say limit exists if right hand limit & left hand limit are equal.

Example: Evaluate $\lim_{x \to 0-} \frac{|x|}{x}$ and $\lim_{x \to 0+} \frac{|x|}{x}$.

Solution: By definition,

$|x| = x$, if $x \ge 0$ and $|x| = -x$, if $x < 0$.

Now $\lim_{x \to 0-} f(x) = \lim_{\substack{x \to 0 \\ (x<0)}} \frac{|x|}{x} = \lim_{x \to 0} \frac{(-x)}{x} = -1$,

$\lim_{x \to 0+} f(x) = \lim_{\substack{x \to 0 \\ (x>0)}} \frac{|x|}{x} = \lim_{x \to 0} \frac{x}{x} = 1$.

Hence $\lim_{x \to 0} \frac{|x|}{x}$ does not exist, since $-1 \ne 1$.

Example: Show that $\lim_{x \to 0} \frac{x - |x|}{x}$ does not exist.

Solution: We have $\lim_{x \to 0-} \frac{x - |x|}{x} = \lim_{x \to 0} \frac{x - (-x)}{x} = \lim_{x \to 0} \frac{2x}{x} = 2$,

and $\lim_{x \to 0+} \frac{x - |x|}{x} = \lim_{x \to 0} \frac{x - x}{x} = 0$.

Hence $\lim_{x \to 0} \frac{x - |x|}{x}$ does not exist.

Example: Evaluate $\lim_{x \to 0} [x + |x|]$.

Solution: $\lim_{x \to 0-} [x + |x|] = \lim_{x \to 0} (x - x) = 0$,

$\lim_{x \to 0+} [x + |x|] = \lim_{x \to 0} (x + x) = \lim_{x \to 0} 2x = 0$.

Hence $\lim_{x \to 0} (x + [x]) = 0$.

5.5.4 Theorems on Limits

1. **(Uniqueness Theorem)**
   Let $S \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$ and f is defined on S.
   Then $\lim_{x \to a} f(x)$ if exist then it is unique.
2. If $\lim_{x \to a} f(x) = L$, then f is bounded on some deleted neighbourhood of $x = a$ i.e., if f is unbounded on some neighbourhood of $c \Rightarrow$ limit at c does not exist.

   For example : $\lim_{x \to 0} \frac{1}{x}$ does not exist.
3. $\lim_{x \to a} f(x)$ and equal to a real number $l \Leftrightarrow$ both left hand limit $\lim_{x \to a^-} f(x)$ and right limit $\lim_{x \to a^+} f(x)$ exist and are equal to l.
4. **Algebra of limits**

Let $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$

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Then

(i) $\lim\limits_{x \to a} \left(f(x) \pm g(x)\right) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = l \pm m$

(ii) $\lim\limits_{x \to a} \left(f(x).g(x)\right) = \lim\limits_{x \to a} \left(f(x)\right). \lim\limits_{x \to a} \left(g(x)\right) = l.m$

(iii) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{l}{m}$ provided $m \neq 0$.

Example:

$$\lim_{x \to 0+} \frac{\sin x}{\sqrt{x}}.$$

Solution:

$$\lim_{x \to 0+} \frac{\sin x}{\sqrt{x}} = \left(\lim_{x \to 0+} \frac{\sin x}{x}\right).\left(\lim_{x \to 0+} \sqrt{x}\right) = 1.0 = 0$$

Example: Evaluate $\lim\limits_{x \to 0} \dfrac{e^{1/x}}{e^{1/x}+1}$.

Solution: Now when $x \to 0^+, 1/x \to \infty, e^{-1/x} \to 0$ and when $x \to 0^-, 1/x \to -\infty, e^{1/x} \to 0$.

$$\therefore \lim_{x \to 0^+} \frac{e^{1/x}}{e^{1/x}+1} = \lim_{x \to 0^+} \frac{1}{1+e^{-1/x}} = 1$$

and $\lim\limits_{x \to 0^-} \dfrac{e^{1/x}}{e^{1/x}+1} = \dfrac{0}{1} = 0$

so that the left hand limit is not equal to the right hand limit.

Hence $\lim\limits_{x \to 0} \dfrac{e^{1/x}}{e^{1/x}+1}$ does not exist.

Example: Find $\lim\limits_{x \to 0} e^x \operatorname{sgn}(x + [x])$, where the signum function is defined as $\operatorname{sgn}(x) = \begin{cases} 1 \text{ if}, & x > 0 \\ 0 \text{ if}, & x = 0 \\ -1 \text{ if}, & x < 0 \end{cases}$ and $[x]$ means the greatest integer $\leq x$.

Solution: Left for the reader.

(Ans. $\lim\limits_{x \to 0} e^x \operatorname{sgn}(x + [x])$ does not exist.)

5. If f and g are defined on a deleted neighbourhood of a point a and $f(x) \geq g(x) \; \forall \; x \in D$ then $\lim\limits_{x \to a} f(x) \geq \lim\limits_{x \to a} g(x)$ provided limit exist.

6. **Sandwich Theorem for Functions:** If function f, g and h are defined on a deleted neighbourhood D of a point a such that $f(x) \geq g(x) \geq h(x) \; \forall \; x \in D$ and $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = l$, then $\lim\limits_{x \to a} g(x)$ exists and equals to l.

7. Let f be strictly increasing on $I \subset \mathbb{R}$ then f^{-1} exist and is strictly increasing on $f(I)$

Reason: Since f is strictly increasing on I, it is one-to-one on I and so f^{-1} exists. To prove that f^{-1} is strictly increasing, let $y_1 < y_2$ be two points of $f(I)$ and let $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. We cannot have $x_1 > x_2$ for this will imply $y_1 > y_2$ contradicting our assumption. Hence the only alternative is $x_1 < x_2$ which means $f^{-1}(y_1) < f^{-1}(y_2)$. Hence f^{-1} is strictly increasing.

8. If f is monotonic on (a,b), then for each c in (a,b) $\lim\limits_{x \to c^+} f(x)$ and $\lim\limits_{x \to c^-} f(x)$ both exist may not be equal.

9. If f is monotonic increasing on (a,b), then for each $c \in (a,b)$

$$\lim_{x \to c^-} f(x) = \sup_{x \in (a,c)} f(x) < f(c) < \inf_{x \in (c,b)} f(x) = \lim_{x \to c^+} f(x)$$

5.5.5 Some Important Limits

(i) $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim\limits_{x \to \infty} \left(1 - \frac{1}{x}\right)^{-x} = e;$

(ii) $\lim\limits_{x \to 0} (1+x)^{1/x} = e;$

(iii) $\lim\limits_{x \to \infty} \left(1 + \frac{y}{x}\right)^x = \lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^{yx} = e^y;$

(iv) $\lim\limits_{x \to 0} \frac{\log(1+x)}{x} = 1;$

(v) $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \log a, \forall a > 0;$

(vi) $\lim\limits_{x \to a} \frac{x^p - a^p}{x - a} = p a^{p-1}, \ \forall p \neq 0$ and $a \neq 0$ if $p = 0;$

(vii) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1;$

(viii) $\lim\limits_{x \to 0} \cos x = 1.$

The above limits are of basic significance.

5.5.6 Infinite Limits and Limits at Infinity

(i) A function f is said to tend to l as $x \to \infty$ if given $\varepsilon > 0$, there exists a positive number k such that $|f(x) - l| < \varepsilon$ whenever $x > k$

Also then we write $\lim\limits_{x \to \infty} f(x) = l$ or $f(x) \to l$ as $x \to \infty$.

(ii) A function f is said to tend to l as $x \to -\infty$ if given $\varepsilon > 0$, there exists a positive number k such that $|f(x) - l| < \varepsilon$ whenever $x < -k$

Also then we write $\lim\limits_{x \to -\infty} f(x) = l$ or $f(x) \to l$ as $x \to -\infty$.

(iii) A function f is said to tend to ∞ as x tends to a, if given $k > 0$, however large, there exists a positive number δ such that $f(x) > k$ whenever $0 < |x - a| < \delta$.

Also then we write $\lim\limits_{x \to a} f(x) = \infty$ or $f(x) \to \infty$ as $x \to a$.

(iv) A function f is said to tend to $-\infty$ as x tends to a, if given $k > 0$, however large, there exists a positive number δ such that $f(x) < -k$ whenever $0 < |x - a| < \delta$.

Also then we write $\lim\limits_{x \to a} f(x) = -\infty$ or $f(x) \to -\infty$ as $x \to a$.

(v) A function f is said to tend to ∞ as $x \to \infty$ if given $k > 0$, however large, there exists a positive number K such that $f(x) > k$ whenever $x > K$

Also then we write $\lim\limits_{x \to \infty} f(x) = \infty$ or $f(x) \to \infty$ as $x \to \infty$.

(vi) A function f is said to tend to $-\infty$ as $x \to \infty$ if given $k > 0$, however large, there exists a positive number K such that $f(x) < -k$ whenever $x > K$

Also then we write $\lim\limits_{x \to \infty} f(x) = -\infty$ or $f(x) \to -\infty$ as $x \to \infty$.

(vii) A function f is said to tend to ∞ as $x \to -\infty$, if given $k > 0$, however large, there exists a positive number K such that $f(x) > k$ whenever $x < -K$.

Also then we write $\lim\limits_{x \to -\infty} f(x) = \infty$ or $f(x) \to \infty$ as $x \to -\infty$.

(viii) A function f is said to tend to $-\infty$ as $x \to -\infty$, if given $k > 0$, however large, there exists a positive number K such that $f(x) < -k$ whenever $x < -K$.

Example: Show that $\lim\limits_{x \to +\infty} \frac{1}{x} = \lim\limits_{x \to -\infty} \frac{1}{x} = 0$.

Solution: Let $\varepsilon > 0$ be given. Choose $M = 1/\varepsilon$. Then

$$x > M \Rightarrow \left|\frac{1}{x}\right| < \frac{1}{M} = \varepsilon \Rightarrow \left|\frac{1}{x} - 0\right| < \varepsilon.$$

Hence $\lim\limits_{x \to +\infty} \frac{1}{x} = 0$,

Now $x < -M \Rightarrow |x| > M \Rightarrow \left|\frac{1}{x}\right| < \frac{1}{M} = \varepsilon$

or $x < -M \Rightarrow \left|\frac{1}{x} - 0\right| < \varepsilon$. Hence $\lim\limits_{x \to -\infty} \frac{1}{x} = 0$.

Examples: Show that

(i) $\lim\limits_{x \to 0+} \frac{1}{x} = +\infty$

(ii) $\lim\limits_{x \to 0-} \frac{1}{x} = -\infty$

(iii) $\lim\limits_{x \to 0} \frac{1}{x^2} = +\infty$

Solutions:

(i) Let $M > 0$ be any large number. Let $\delta = 1/M$.

Then $0 < x < \delta \Rightarrow \frac{1}{x} > \frac{1}{\delta} \Rightarrow \frac{1}{x} > M$.

Hence $\lim\limits_{x\to 0+} \frac{1}{x} = +\infty$,

(ii) Again $-\delta < x < 0 \Rightarrow -\frac{1}{\delta} > \frac{1}{x} \Rightarrow -M > \frac{1}{x} \Rightarrow \frac{1}{x} < -M$.

Hence $\lim\limits_{x\to 0-} \frac{1}{x} = -\infty$,

Note : $\lim\limits_{x\to 0} \frac{1}{x}$ does not exist.

(iii) Now $-\delta < x < \delta \Rightarrow \frac{1}{x^2} > \delta \Rightarrow \frac{1}{x^2} > M$.

Hence $\lim\limits_{x\to 0} \frac{1}{x^2} = +\infty$,

Example: Find $\lim\limits_{x\to\infty} \frac{x^2-4}{x^2+7x+12}$.

Solution: Dividing numerator and denominator by x^2, we get

$$\lim_{x\to\infty} \frac{x^2-4}{x^2+7x+12} = \lim_{x\to\infty} \frac{1-4/x^2}{1+7/x+12/x^2}$$

$$= \frac{1-4\lim\limits_{x\to\infty}\left(1/x^2\right)}{1+7\lim\limits_{x\to\infty}\left(1/x\right)+12\lim\limits_{x\to\infty}\left(1/x^2\right)}$$

$$= \frac{1-0}{1+0+0} = 1.$$

Example: Evaluate $\lim\limits_{x\to 0} \frac{e^{1/x}-1}{e^{1/x}+1}$.

Solution: We know $e^t = 1+t+\frac{t^2}{2!}+\ldots$

It follows that $e^t \to \infty$ as $t \to +\infty$. ...(1)

By previous example, $x \to 0+ \Rightarrow 1/x \to +\infty \quad \Rightarrow e^{1/x} \to +\infty$, by (1)

$\Rightarrow \frac{1}{e^{1/x}} \to 0 \quad \Rightarrow e^{-1/x} \to 0$

Again $x \to 0- \Rightarrow -\frac{1}{x} \to +\infty \quad \Rightarrow e^{-1/x} \to +\infty$, by (1)

$\Rightarrow \frac{1}{e^{1/x}} \to +\infty \quad \Rightarrow e^{1/x} \to 0$.

Hence $x \to 0+ \Rightarrow e^{-1/x} \to 0$ and $x \to 0- \Rightarrow e^{1/x} \to 0$. (A)

We have $\lim\limits_{x\to 0-} \frac{e^{1/x}-1}{e^{1/x}+1} = \frac{0-1}{0+1} = -1$, using (A) and $\lim\limits_{x\to 0+}\dfrac{e^{1/x}-1}{e^{1/x}+1}=\lim\limits_{x\to 0+}\dfrac{1-e^{-1/x}}{1+e^{-1/x}}=\dfrac{1-0}{1+0}=1$, by (A).

Hence $\lim\limits_{x\to 0}\dfrac{e^{1/x}-1}{e^{1/x}+1}$ does not exist. $(\because -1\neq 1)$

Example: Show that $\lim\limits_{x\to 0}\dfrac{xe^{1/x}}{1+e^{1/x}}=0$.

Solution: We know that $x\to 0+\Rightarrow e^{-1/x}\to 0$,

and $x\to 0-\Rightarrow e^{1/x}\to 0$.

We have $\lim\limits_{x\to 0-}\dfrac{xe^{1/x}}{1+e^{1/x}}=\dfrac{0.0}{1+0}=0$,

and $\lim\limits_{x\to 0+}\dfrac{xe^{1/x}}{1+e^{1/x}}=\lim\limits_{x\to 0+}\dfrac{x}{e^{-1/x}+1}=\dfrac{0}{0+1}=0$.

Hence $\lim\limits_{x\to 0}\dfrac{xe^{1/x}}{1+e^{1/x}}=0$.

Example: Prove that $\lim\limits_{x\to\infty}\dfrac{1}{x+1}=0$.

Solution: Let $f(x)=1/(x+1)$ and let $\varepsilon>0$ be given. Then

$|f(x)-0|=\dfrac{1}{x+1}<\dfrac{1}{x}<\varepsilon$ if $x>\dfrac{1}{\varepsilon}$

Taking $k=1/\varepsilon$. Then, we see that

$\left|\dfrac{1}{x+1}-0\right|<\varepsilon$ whenever $x>k\Rightarrow\ \lim\limits_{x\to\infty}\dfrac{1}{x+1}=0$.

Example: Evaluate $\lim\limits_{x\to 0}\dfrac{1}{|x|}$

Solution: Let $f(x)=1/|x|$. Let $k>0$ be given and let $\delta=1/k$. Then

$0<|x-0|<\delta\Rightarrow|x|<\delta\Rightarrow|x|<1/k\Rightarrow\dfrac{1}{|x|}>k$

Thus, $f(x)>k$ whenever $0<|x-0|<\delta\Rightarrow\lim\limits_{x\to 0}\dfrac{1}{|x|}=\infty$

Example: Find $\lim\limits_{x\to 0}[x]$, where $[x]$ denotes the greatest integer not greater than x.

Solution: Right-hand limit $=\lim\limits_{x\to 0+0}[x]=\lim\limits_{h\to 0+}[0+h]=0$

and Left-hand limit $=\lim\limits_{x\to 0-0}[x]=\lim\limits_{h\to 0+}[0-h]=-1$

Since right-hand limit $\neq$ left-hand limit, so $\lim\limits_{x\to 0}[x]$ does not exist

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CHAPTER 6

CONTINUITY

6.1. Definitions:

Let $S \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$ and $f : S \to \mathbb{R}$ be a real valued function. Then we say f is continuous at $\alpha \in S$ if any of the following condition is satisfied:

(i) $\alpha \in S - S'$ (i.e. α is an isolated point of S)

(ii) $\alpha \in S'$ and $\lim_{x \to \alpha} f(x) = f(\alpha)$ (i.e. limit of f exist at $x = \alpha$ and equal to the value of the function)

Remarks:

(a) If any of the condition fails at a point then the function f is said to be discontinuous at that point and the point is called a point of discontinuity of f.

(b) If limit point of S is not member of S. Then f is assumed to be discontinuous at that point.

Example: Let $f : \mathbb{R} - \{1\} \to \mathbb{R}$ be defined as $f(x) = \sin x$. Then at $x = 1$, f is discontinuous function.

(c) At isolated points, $f(x)$ is said to be continuous if it is defined there at.

Example: Let $f : [0,1] \cup \{3\} \to \mathbb{R}$ defined by $f(x) = \sin x$. Then f is continuous at $x = 3$.

Example: The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$, is continuous at $x = 0$.

Solution: We know, $\lim_{x \to 0} f(x) = \lim_{x \to 0}\left(x \sin\frac{1}{x}\right) = 0$. Also, $f(0) = 0$.

So $\lim_{x \to 0} f(x) = f(0)$.

Hence, f is continuous at $x = 0$.

Example: The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} \dfrac{xe^{1/x}}{1+e^{1/x}}, & x \neq 0 \\ 0 & x = 0 \end{cases}$, is continuous at $x = 0$.

Solution: We have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{xe^{1/x}}{1+e^{1/x}} = 0$

$\therefore \lim_{x \to 0} f(x) = f(0) \quad (\because f(0) = 0)$

Hence, f is continuous at $x = 0$.

Example: The function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\begin{cases}\dfrac{x}{1+e^{1/x}}, & x\neq 0\\ 0, & x=0\end{cases}$, is continuous at $x=0$.

Solution: We have, $\lim\limits_{x\to 0-} f(x)=\lim\limits_{x\to 0-}\dfrac{x}{1+e^{1/x}}=\dfrac{0}{1+0}=0$, $\quad\left(\because \lim\limits_{x\to 0-} e^{1/x}=0\right)$

$$\lim_{x\to 0+} f(x)=\lim_{x\to 0+}\frac{x}{1+e^{1/x}}=\lim_{x\to 0+}\frac{xe^{-1/x}}{e^{-1/x}+1}=\frac{0.0}{0+1}=0. \qquad \left(\because \lim_{x\to 0+} e^{-1/x}=0\right)$$

Also, $f(0)=0$. Hence f is continuous at $x=0$.

Example: Let f be the function defined on $\mathbb{R}$ by $f(x)=[x]$, for all $x\in\mathbb{R}$,where $[x]$ denote the greatest integer not exceeding x. Then f is discontinuous at the points $x=0,\pm1,\pm2,\pm3,...$ and is continuous at every other point.

Example: Let f be the function defined on $\mathbb{R}$ as $f(x)=|x|+[x]$, for all $x\in\mathbb{R}$ .Determine the points of discontinuity of f .

Solution: We have,

$f(x)=x+0=x$, for $0\le x<1$ $f(x)=x+1$, for $1\le x<2$;

$f(x)=x+2$, for $2\le x<3$; $f(x)=x+3$, for $3\le x<4$;

...

...

$f(x)=-x-1$, for $-1\le x<0$; $f(x)=-x-2$, for $-2\le x<-1$;

and so on.

We have

$\lim\limits_{x\to 0-} f(x)=\lim\limits_{x\to 0}(-x-1)=-1$, $\lim\limits_{x\to 0+} f(x)=\lim\limits_{x\to 0} x=0$;

$\lim\limits_{x\to 1-} f(x)=\lim\limits_{x\to 1} x=1$, $\lim\limits_{x\to 1+} f(x)=\lim\limits_{x\to 1}(x+1)=2$;

$\lim\limits_{x\to 2-} f(x)=\lim\limits_{x\to 2}(x+1)=3$, $\lim\limits_{x\to 2+} f(x)=\lim\limits_{x\to 2}(x+2)=4$;

$\lim\limits_{x\to -1-} f(x)=\lim\limits_{x\to -1}(-x-2)=-3$, $\lim\limits_{x\to -1+} f(x)=\lim\limits_{x\to -1}(-x-2)=-2$;

and so on.

Thus $\lim\limits_{x\to n-} f(x)\neq \lim\limits_{x\to n+} f(x)\ \forall n\in\mathbb{Z}$.

Hence the points of discontinuity of f are $0,\pm1,\pm2,\pm3,...$

6.1.1. Second Definition of Continuity

Let $S\subseteq\mathbb{R}$ and $f:S\to\mathbb{R}$ be a function then we say f is continuous at the point $\alpha\in\overline{S}\Leftrightarrow$ for every sequence $\langle x_n\rangle$ in S such that $\langle x_n\rangle\to\alpha\Rightarrow f(x_n)\to f(\alpha)$

Remarks:

(i) If we can find two sequences $\langle a_n \rangle, \langle b_n \rangle$ in S such that $\langle a_n \rangle, \langle b_n \rangle$ converges to α but $f(a_n), f(b_n)$ converges to different limits. Then f is not continuous at $x = \alpha$.

(ii) If we can find a sequence $\langle a_n \rangle$ which converges to α but $f(a_n)$ does not converges at all. Then f is not continuous at $x = \alpha$.

Examples related to above definitions:

(a) Let $f:[0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{q} & x = \dfrac{p}{q} \in [0,1],\ p,q \in \mathbb{Z}^+ \\ 0 & \text{otherwise} \end{cases}$$

Then f is continuous at every irrational number in $[0,1]$ and 0 but discontinuous at all rationals except 0.

(b) Let $f:[-1,1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x & x \in [-1,1] \cap \mathbb{Q}^c \text{ and } x = 0 \\ \sqrt{\dfrac{1+p^2}{1+q^2}} & x = \dfrac{p}{q} \text{in } [-1,1] \end{cases}$$

Then limit does not exist on $[-1,0)$, so not continuous on $[-1,0)$

Hence, f is continuous on $(0,1) \cap \mathbb{Q}^c$ and at $x = 0,1$

(c) Let $f:\mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{Q}^c \end{cases}$$

Then, f is not continuous at any real number.

(d) Let $f:[0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} p\sin\left(\dfrac{1}{q}\right) & x = \dfrac{p}{q} \text{ in } [0,1],\ p,q \in \mathbb{Z}^+ \\ x & x = 0 \text{ or } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

Then f is continuous at all irrational in $[0,1]$ & 0 and not continuous at all rationals except 0.

Note: Let $f:\mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \phi_1(x), & x \in S \\ \phi_2(x), & x \notin S \end{cases}$$

Where $\phi_1(x), \phi_2(x)$ are continuous function on their domain and S is any dense subset of $\mathbb{R}$.

Then roots of the equation $\phi_1(x) - \phi_2(x) = 0$ are only point of continuity of f.

Also if the roots are repeated then f is differentiable at that point also.

Example:

(a) Let $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}x^6-1 & x\in\mathbb{Q}\\ 1-x^6 & x\in\mathbb{Q}^c\end{cases}$$

Then f is continuous at $x=1,-1$ only.

Reason: The equation $\left(x^6-1\right)-\left(1-x^6\right)=0$ gives $x^6=1$

Which have only two distinct real roots viz. 1 and -1.

(b) Let $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}x & x\in\mathbb{Q}\\ -x & x\in\mathbb{Q}^c\end{cases}$$

Then f is continuous only at $x=0$.

6.1.2. Third Definition of Continuity

A function $f:S\to\mathbb{R}$ is continuous at a point $c\in S$ iff for given any $\varepsilon-$ neighbourhood $V_\varepsilon(f(c))$ of $f(c)$ there exist a $\delta-$neighbourhood $V_\delta(c)$ of c such that $f(S\cap V_\delta(c))\subseteq V_\varepsilon(f(c))$ i.e. if x is any point of $S\cap V_\delta(c)$, then $f(x)$ belongs to $V_\varepsilon(f(c))$.

6.2. Types of Discontinuity

1. **Removable discontinuity (simple discontinuity):** Let $S\subseteq\mathbb{R}$ and $f:S\to\mathbb{R}$ be a function. If $\lim\limits_{x\to\alpha}f(x)$ exist but either $f(x)$ is not defined at $x=\alpha$ or it is defined and $\lim\limits_{x\to\alpha}f(x)\neq f(\alpha)$. Then we say that the function has removable discontinuity at $x=\alpha$.

Example:

(a) Let $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}\dfrac{\sin x}{x} & x\neq 0\\ 0 & x=0\end{cases}$$

Then f has removable type discontinuity at $x=0$.

Reason: $\lim\limits_{x\to0}f(x)=\lim\limits_{x\to0}\left(\dfrac{\sin x}{x}\right)=1$ and $f(0)=0$.

(b) Let $f:(0,1)\to\mathbb{R}$ defined by

$$f(x)=x \qquad \forall x\in(0,1).$$

Then f is discontinuous at $x=0$ and 1.

Note: This type of discontinuity can be removed by defining a new function g as

$$g(x)=\begin{cases}f(x) & \text{when } x\neq\alpha\\ \lim\limits_{x\to\alpha}f(x) & \text{when } x=\alpha\end{cases}$$

Then g is continuous at $x=\alpha$.

Put Your Own Notes

2. **Discontinuity of first kind or (Jump discontinuity):** A function $f : S \to \mathbb{R}$ is said to have discontinuity of first kind (or Jump discontinuity) at $x = \alpha$ if at this point left hand limit (L.H.L.) and right had limit (R.H.L.) both exist but not equal furthermore.

   (i) A function f is said to have a discontinuity of the first kind from the left to α if L.H.L exist but it is not equal to $f(\alpha)$.

   (ii) A function f is said to have a discontinuity of first kind from the right to α if R.H.L. exist but it is not equal to $f(\alpha)$.

   Note: Jump at $(x = \alpha) = \left| \lim\limits_{x \to \alpha^-} f(x) - \lim\limits_{x \to \alpha^+} f(x) \right|$

   Example:

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x]\ \forall x \in \mathbb{R}$

   Then f has jump kind of discontinuity at every $\alpha \in \mathbb{Z}$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \{x\}\ \forall x \in \mathbb{R}$, where $\{x\}$ is fractional part of x.

   Then at $x = 1$ f has jump kind of discontinuity.

   (c) Let $f : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

   Then at $x = 0$, f has jump kind of discontinuity.

3. **Discontinuity of second kind (Mixed discontinuity):** A function $f(x)$ is said to have a discontinuity of second kind (or mixed discontinuity) at $x = \alpha$, if any of Right hand limit or Left hand limit or both does not exist at $x = \alpha$.

   Examples:

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{Q}^c \end{cases}$$

   Then for any $\alpha \in \mathbb{R}$.

   $\exists\ \delta > 0$ and $x_1 \in \mathbb{Q} \cap (\alpha, \alpha + \delta)$, $x_2 \in \mathbb{Q}^c \cap (\alpha, \alpha + \delta)$ such that $|f(x_1) - f(x_2)| = 2$

   $\Rightarrow$ R.H.L. does not exist

   $\Rightarrow$ f has discontinuity of second kind at every $\alpha \in \mathbb{R}$.

   (b) Let $f : [0,1] \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   Then at $x = 0$ L.H.L. and R.H.L. both does not exist

   $\Rightarrow$ f has a discontinuity of second kind at $x = 0$.

(c) Let $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}\dfrac{e^{1/x}\sin(1/x)}{1+e^{1/x}} & x\neq 0\\ 0 & x=0\end{cases}$$

Then f has discontinuity of second kind at $x=0$.

Reason: We know that

$x\to 0-\Rightarrow e^{1/x}\to 0$ and $x\to 0+\Rightarrow e^{-1/x}\to 0$.

$$\therefore \lim_{x\to 0+}\frac{e^{1/x}\sin(1/x)}{1+e^{1/x}}=\lim_{x\to 0+}\frac{\sin(1/x)}{e^{-1/x}+1}=\lim_{x\to 0+}\sin\frac{1}{x},$$

which does not exist.

Hence, f has a discontinuity of the second kind from the right at $x=0$.

6.3. Algebra of Continuity

Let $S\subseteq\mathbb{R}$ and $f\,\&\,g$ real valued be functions on S. Let $a,b\in\mathbb{R}$ suppose that $\alpha\in S$ and $f\,\&\,g$ are continuous at α. Then

(i) Linear combination of f and g is also continuous at $x=\alpha$ i.e. $af+bg$ is continuous at $x=\alpha$.

(ii) Product of f and g is continuous at α i.e. $f.g$ is continuous at $x=\alpha$.

(iii) $\dfrac{f}{g}$ (provided $g(x)\neq 0\ \forall x\in S$) is continuous at $x=\alpha$.

(iv) $|f|$ is also continuous at $x=\alpha$.

(v) $\max\{f,g\}=\dfrac{(f+g)+|f-g|}{2}$ is also continuous at $x=\alpha$.

(vi) $\max\{f,g\}=\dfrac{(f+g)-|f-g|}{2}$ is also continuous at $x=\alpha$.

Remark:

(a) Converse of the above results may not be true.

Example: Let $f,g:\mathbb{R}\to\mathbb{R}$ be functions defined as $f(x)=\begin{cases}1 & x\in\mathbb{Q}\\ -1 & x\notin\mathbb{Q}\end{cases}$

and $g(x)=\begin{cases}-1 & x\in\mathbb{Q}\\ 1 & x\notin\mathbb{Q}\end{cases}$

Clearly both f and g are discontinuous at each real number.

But $(f+g)(x)=0\ \ \forall x\in\mathbb{R}$ is continuous function

$(f.g)(x)=-1\ \ \forall x\in\mathbb{R}$ is continuous function

$\left(\dfrac{f}{g}\right)(x)=-1\ \ \forall x\in\mathbb{R}$ is continuous function.

(b) If f is continuous function at $x=\alpha$ and g is discontinuous function at $x=\alpha$ then $f\pm g$ cannot be continuous function at $x=\alpha$.

(c) If f and g are two continuous functions at $x=\alpha$ such that gof and fog is defined. Then fog and gof both are continuous at $x=\alpha$. Converse need not be true.

Put Your Own Notes

Example: Consider two functions f and g defined as $f(x)=\begin{cases}1 & x\in\mathbb{Q}\\ -1 & x\in\mathbb{Q}^c\end{cases}$

and $g(x)=\begin{cases}-1 & x\in\mathbb{Q}\\ 1 & x\in\mathbb{Q}^c\end{cases}$

Then f and g are nowhere continuous but fog and gof both are continuous on $\mathbb{R}$.

6.4. Some Important Theorems on Continuous Functions

1. **Boundedness Theorem:** Let $I=[a,b]$ and $f:I\to\mathbb{R}$ be a continuous function, then f is bounded on I.

   Remark: Each hypothesis of the Boundedness Theorem is needed, we can construct examples which show that the conclusion may fails if any one of the hypothesis is relaxed.

   (a) The interval must be bounded: The function $f(x)=x$ for x in the unbounded interval $A=[0,\infty)$ is continuous but not bounded on A.

   (b) The interval must be closed: The function $g(x)=\frac{1}{x}$ for x in the half-open interval $B=(0,1]$ is continuous but not bounded on B.

   (c) The function must be continuous: The function $h(x)$ defined on the closed interval $C=[0,1]$ by $h(x)=\frac{1}{x}$ for $x\in(0,1]$ and $h(0)=1$ is discontinuous and unbounded on C.

   (d) Converse of the theorem need not be true.

   Example: Let $f:[0,1]\to\mathbb{R}$ defined by $f(x)=\begin{cases}1 & x\in\mathbb{Q}\cap[0,1]\\ -1 & x\in\mathbb{Q}^c\cap[0,1]\end{cases}$

   Then f is bounded but not continuous.

   Definitions:

   (i) **Absolute Maximum:** Let $A\subseteq\mathbb{R}$ and $f:A\to\mathbb{R}$ be a function, then we say that f has an **Absolute Maximum** on A if there is a point $x^*\in A$ such that $f(x^*)\geq f(x)$ for all $x\in A$.

   (ii) **Absolute Minimum:** Let $A\subseteq\mathbb{R}$ and $f:A\to\mathbb{R}$ be a function, then we say that f has an **Absolute Minimum** on A if there is a point $x_*\in A$ such that $f(x_*)\leq f(x)$ for all $x\in A$.

   We say that x^* is an **absolute maximum point** for f on A and that x_* is an **absolute minimum point** for f on A, if they exist.

2. **The Maximum-Minimum Theorem:** Let $I=[a,b]$ and $f:I\to\mathbb{R}$ be continuous on I, then f has an absolute maximum and an absolute minimum on I.

Remark:

(a) If I is not closed interval then result may not hold.

Example: Let $f:(0,1)\to\mathbb{R}$ defined by $f(x)=\frac{1}{x}$ is continuous on $(0,1)$, but does not have absolute maximum on I.

(b) If I is not bounded interval then result may not hold.

Example: Let $f:[0,\infty)\to\mathbb{R}$ defined by $f(x)=x \quad \forall\, x\in[0,\infty)$

Then f is continuous on $I=[0,\infty)$ but does not have absolute maximum on I.

(c) Condition of continuity of f on closed and bounded interval cannot be relaxed.

Example: Let $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}1 & x=0\\ \frac{1}{x} & x\neq 0\end{cases}$$

Then f is discontinuous and unbounded on $I=[0,1]$

3. **The Location of Roots Theorem:** Let $I=[a,b]$ and $f:I\to\mathbb{R}$ be continuous on I and if $f(a)<0<f(b)$ or $f(a)>0>f(b)$ i.e. $f(a).f(b)<0$ then there exist a number $c\in(a,b)$ such that $f(c)=0$.

Remark:

(a) The condition $f(a).f(b)<0$ cannot be relaxed.

Example: Let $f:[1,2]\to\mathbb{R}$ defined by $f(x)=x$

Then f is continuous on $I=[1,2]$ but there does not exist any $c\in(1,2)$ such that $f(c)=0$.

(b) The condition of continuity of f cannot be relaxed.

Example: Let $f:[-1,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}-1 & x\leq 0\\ 1 & x>0\end{cases}$$

Then $f(-1).f(1)<0$, but f does not vanish anywhere in $[-1,1]$

(c) If $I=[a,b]$ and $f:I\to\mathbb{R}$ be such that $f(a).f(b)<0$ and there exist $c\in(a,b)$ such that $f(c)=0$. Then f may not be continuous.

Example: Let $f:[-1,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}-1 & -1\leq x<0\\ 0 & x=0\\ 1 & 0<x\leq 1\end{cases}$$

(d) The converse of the above theorem need not be true i.e. Let $I=[a,b]$ and $f:I\to\mathbb{R}$ is continuous on I and if $\exists\ c\in(a,b)$ such that $f(c)=0$ then $f(a).f(b)$ need not be negative.

Example: Let $f:[-1,1]\to\mathbb{R}$ defined by $f(x)=x^2\ \ \forall\, x\in[-1,1]$.

(e) If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and there exist $\alpha,\beta\in[a,b]$ such that $f(\alpha)f(\beta)<0$ then there exist $c\in(\alpha,\beta)$ such that $f(c)=0$

4. **Bolzano's Intermediate Value Theorem (I.V.P.):**

Let I be an interval and $f:I\to\mathbb{R}$ be continuous on I and if there exist $a,b\in I$ and $k\in\mathbb{R}$ satisfying $f(a)<k<f(b)$, then there exists a point $c\in I$ between a and b such that $f(c)=k$.

Remark:

(a) The condition of continuity of f cannot be relaxed.

Example: Let $f:[0,1]\to\mathbb{R}$ be defined as

$$f(x)=\begin{cases} x & x\in(0,1)\\ 1 & x=0\\ 0 & x=1\end{cases}$$

Then f is continuous on $(0,1)$ but not continuous at 0 and 1.

Also, $f\left(\frac{1}{2}\right)=\frac{1}{2}<\frac{3}{4}<f(0)=1$

But there does not exist any $c\in\left(0,\frac{1}{2}\right)$ such that $f(c)=\frac{3}{4}$

(b) Continuity of f on $[a,b]$ is sufficient condition for **I.V.P.** i.e. If a function f satisfy the **I.V.P.** on $[a,b]$ then it may not be continuous on $[a,b]$.

Example: Let $f:\left[\frac{-2}{\pi},\frac{2}{\pi}\right]\to\mathbb{R}$ be defined as

$f(x)=\begin{cases}\sin\left(\frac{1}{x}\right) & x\neq 0\\ 0 & x=0\end{cases}$. Then satisfy $I.V.P$ but it is not continuous on $\left[\frac{-2}{\pi},\frac{2}{\pi}\right]$

(c) **I.V.P.** is a necessary condition for continuity of a function f on $[a,b]$ i.e. if a function f defined on $[a,b]$ does not satisfy **I.V.P.** on $[a,b]$, then it cannot be continuous on $[a,b]$

5. **Preservation of Interval Theorem :**

1. Let I be a closed and bounded interval and let $f:I\to\mathbb{R}$ be continuous on I. Then the set $S=\{f(x):x\in I\}$ is closed and bounded interval.

i.e., image of a closed and bounded interval under continuous map is closed and bounded interval.

Warning: If $I:=[a,b]$ is an interval and $f:I\to\mathbb{R}$ is continuous on I, we have proved that $f(I)$ is the interval $[m,M]$. We have not proved (and it is not always true) that $f(I)$ is the interval $[f(a),f(b)]$.

2. Let I be an interval and let $f:I\to\mathbb{R}$ be continuous on I. Then the set $f(I)$ is an interval.

Remark:

(a) Condition of closeness of I cannot be relaxed.

Example: Let $f:(0,1)\to\mathbb{R}$ be defined as

$$f(x)=\frac{1}{x}\ \forall\, x\in(0,1)$$

Then f is continuous on $(0,1)$

But $S=\{f(x):x\in(0,1)\}$ is $(0,\infty)$ which is neither bounded nor closed.

(b) Condition of boundedness cannot be relaxed.

Example: Let $f:[0,\infty]\to\mathbb{R}$ be defined as

$$f(x)=x\quad\forall\, x\in[0,\infty)$$

Then f is continuous on $I=[0,\infty)$ but $S=\{f(x):x\in I\}$ is $[0,\infty)$, which is unbounded.

(c) Image of closed and bounded interval under continuous map f is closed and bounded interval and the interval is $[m,M]$, where $m=\inf\{f(x):x\in I\}$, $M=\sup\{f(x):x\in I\}$

6. **Sign Retainment Property of a Continuous function**

If a function f is continuous at an interior point c of an interval $[a,b]$ and $f(c)\neq 0$ then $\exists$ a $\delta>0$ such that $f(x)$ has the same sign as $f(c)$, for every $x\in(c-\delta,c+\delta)$

Remark:

(a) If f is continuous at the end point b of $[a,b]$ and $f(b)\neq 0$ then there exist an interval $(b-\delta,b)$ such that $f(x)$ has the sign of $f(b)$ for all x in $(b-\delta,b]$

(b) A similar result hold for continuity at point a.

(c) This result can be extended for any subset S of $\mathbb{R}$.

i.e., If $S\subseteq\mathbb{R}$, and $f:S\to\mathbb{R}$ is continuous function at $x=\alpha$ and $f(\alpha)\neq 0$.

Then there exist $\delta>0$ such that $f(x)$ has the sign of $f(\alpha)$ for all x in $(\alpha-\delta,\alpha+\delta)\cap S$

7. **Identity Theorem**

(First Form)

Let f be a continuous function on an interval I and $f(x)=0\ \forall\ x\in I\cap\mathbb{Q}$. Then $f(x)\equiv 0\ \forall\ x\in I$

(Second Form)

If $f(x)$ and $g(x)$ are continuous on an interval I and $f(x)=g(x)\forall\, x\in I\cap\mathbb{Q}$. Then $f(x)=g(x)\forall\ x\in I$

Generalized form: Let f and g are two continuous functions on $S\subseteq\mathbb{R}$ and $f(x)=g(x)\ \forall x\in T$ (where T is a dense subset of S). Then $f(x)=g(x)\ \forall\ x\in S$.

Remarks:

(a) If f is monotonic on I. Then $\lim\limits_{x\to\alpha^-} f(x)$ and $\lim\limits_{x\to\alpha^+} f(x)$ both exist may not be equal.

i.e., A monotonic function can have only jump kind of discontinuity

(b) If f is monotonic and satisfy $I.V.P$ on I then it is continuous function.

(c) The set of point of discontinuity of the first kind of real valued function is countable.

(d) There is no real valued function which is continuous on every rational point and discontinuous at every irrational point.

Some Important Examples:

1. Let f be a function defined on $\mathbb{R}$ which satisfy the condition $|f(x)|\le|x|\ \forall\ x\in\mathbb{R}$

Then f is continuous at $x=0$.

Reason: Since $0\le|f(0)|\le|0|$

$\Rightarrow f(0)=0$

Let $\varepsilon>0$ be given.

Consider,

$|f(x)-f(0)|=|f(x)|\le|x|=|x-0|<\varepsilon$

Take $\delta=\varepsilon$. Then, $|f(x)-f(0)|<\varepsilon$, whenever $|x-0|<\delta$

Thus, f is continuous at $x=0$.

2. Let f be a function defined on $\mathbb{R}$ which satisfy the condition $f(x+y)=f(x)+f(y)\ \forall\ x,y\in\mathbb{R}$

Then, if f is continuous at a point c then f is continuous at all points of $\mathbb{R}$

3. Let $f:\mathbb{R}\to\mathbb{R}$ be a function which satisfy the property

(i) $|f(x)|\le|x|\ \forall\ x\in\mathbb{R}$

(ii) $f(x+y)=f(x)+f(y)\ \forall\ x,y\in\mathbb{R}$

Then $f(x)=cx\ \forall x\in\mathbb{R}$ where $c=f(1)$

4. Let $f:\mathbb{R}\to\mathbb{R}$ be a function satisfying the condition

(i) $|f(x)|\le|x|\ \forall\ x\in\mathbb{R}$

(ii) $f(x+y)=f(x)f(y),\ \ \forall\ x,y\in\mathbb{R}$

Then, $f(x)=c^x\ \ \forall x\in\mathbb{R}$, where $c=f(1)$

5. If f is a continuous on $\mathbb{R}$ and any of the following conditions are satisfied then f must be constant function.

(i) $f(x)=f(mx)\forall x\in\mathbb{R},|m|\ne1,m\in\mathbb{R}$

(ii) $f(x)=f(2x+1)\ \forall\ x\in\mathbb{R}$

(iii) $f(x)=f\left(x^2\right)\forall\ x\in\mathbb{R}$.

Fixed Point: Let $S\subseteq\mathbb{R}$ and $f:S\to\mathbb{R}$ be a any function. Then $\alpha\in S$ is said to be a fixed point of f if $f(\alpha)=\alpha$

Remark:

(a) Fixed point of $y=f(x)$ are the point of intersection of the curve $y=f(x)$ & $y=x$.

(b) If $h(x)=f(x)-x$, then the zeroes of h are the fixed point of f *i.e.*, number of zeros of h is equal to numbers of fixed point of f.

Example:

(a) The function $f:\mathbb{R}\sim\{0\}\to\mathbb{R}$ defined by $f(x)=2x+\frac{1}{x}-\sin x$ has no fixed point

Reason: Define a function h as

$h(x)=f(x)-x=x+\frac{1}{x}-\sin x$

Then h has no zero, since Arithmetic Mean ≥ Geometric Mean

$$\Rightarrow\quad \frac{x+\frac{1}{x}}{2}\ge\sqrt{x.\frac{1}{x}}$$

$$\Rightarrow\quad x+\frac{1}{x}\ge2$$

$$\Rightarrow\quad x+\frac{1}{x}-\sin x\ge2-\sin x\ge1$$

$$\Rightarrow\quad x+\frac{1}{x}-\sin x\ne0\ \ \forall\ x$$

$$\Rightarrow\quad x+\frac{1}{x}\ne\sin x\ \ \forall\ x$$

Thus f has no fixed point.

(b) Let $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x+\sin x$

Let $A=\{x\in\mathbb{R}/f(x)=x\}$. Then card (A) is countably infinite

Reason: Let us define a function h as

$h(x)=f(x)-x$

$=\sin x+x-x$

$=\sin x$

Then $h(x)=0\ \forall\ x=n\pi$ where $n\in\mathbb{Z}$

Thus, card A is countably infinite.

(c) Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined as $f(x)=e^x\ \forall\ x\in\mathbb{R}$. Then f has no fixed point.

Reason: Since the line $y=x$ and $y=f(x)$ does not intersect anywhere.

(d) Let $f:\mathbb{R}-\{0\}\to\mathbb{R}$ be a function defined as $f(x)=|\log_2|x||\ \forall\ x\in\mathbb{R}-\{0\}$.

Then f has 1 fixed point.

Reason: Since the line $y=x$, and $y=f(x)$ intersect at exactly 1 points.

8. Fixed Point Existence Theorem

If $f:[a,b]\to[a,b]$ is a continuous function. Then f has a fixed point. *i.e.*, $\exists\ x_0\in[a,b]$ such that $f(x_0)=x_0$

Remark:

(a) Condition of closeness cannot be relaxed.

Example: Let $f:(0,1)\to(0,1)$ be a function defined by $f(x)=x^2$. Then f is continuous on $(0,1)$, but it does not have any fixed point.

(b) Condition of boundedness of the interval cannot be relaxed.

Example: Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(x)=e^x$. Then f is continuous on $\mathbb{R}$ but f does not have any fixed point.

(c) If domain and range are not equal then function may not have fixed point

Example: Let $f:(0,1)\to(0,\infty)$ be a function defined by $f(x)=\frac{1}{x}$.

Then f is continuous on $(0,1)$ but does not have any fixed point.

(d) If $f:[a,b]\to[c,d]$ is a continuous function where $[c,d]\subseteq[a,b]$. Then f must have a fixed point.

Reason: Take the restriction of f on $[c,d]$.

(e) This theorem gives only the existence of a fixed point. i.e., if $f:[a,b]\to[a,b]$ is a continuous function then f may have countably infinite, finite uncountable numbers of fixed point.

(i) Let $f:[-1,1]\to[-1,1]$ be a function defined as $f(x)=\begin{cases} x\sin\left(\dfrac{1}{x}\right) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$ has countably infinite fixed points.

(ii) Let $f:[-1,1]\to[-1,1]$ be a function defined as $f(x)=x^2$ has two fixed point.

(iii) Let $f:[-1,1]\to[-1,1]$ be a function defined as $f(x)=x$ has uncountable numbers of fixed point.

Result: If f is differentiable on $\mathbb{R}$ & $|f'(x)|\leq\alpha<1$ (where $f'(x)$ denote the derivative of f), then f has at most one fixed point.

6.5. Uniform Continuity

A function f defined on an interval I is said to be uniformly continuous on I if for each $\varepsilon>0$ there exists a $\delta>0$ such that

$|f(x_2)-f(x_1)|<\varepsilon$, for arbitrary points x_1,x_2 of I for which $|x_1-x_2|<\delta$

Note: The uniform continuity of f on arbitrary set S can be defined by replacing the interval I by S.

Example: The function $f(x)=x/(x+1)$ is uniformly continuous for $x\in[0,2]$

Reason: Let x,y be two arbitrary points in $[0,2]$. Then $x\geq 0, y\geq 0$.

$\Rightarrow x+1\geq 1$ and $y+1\geq 1$

$$\Rightarrow (x+1)(y+1)\geq 1 \qquad ...(1)$$

Now, $$|f(x)-f(y)|=\left|\frac{x}{x+1}-\frac{y}{y+1}\right|=\frac{|x-y|}{(x+1)(y+1)}\leq|x-y| \text{ by (1)}$$

Let $\varepsilon>0$ be given. Taking $\delta=\varepsilon$, we get

$|f(x)-f(y)|<\varepsilon$, whenever $|x-y|<\delta\ \forall x,y\in[0,2]$

Hence, f is uniformly continuous in $[0,2]$.

Example: Let $f(x)=x^2, x\in\mathbb{R}$. Then f is uniformly continuous on every closed and finite interval but is not uniformly continuous on $\mathbb{R}$.

Example: The function $f(x)=1/x^2$ is uniformly continuous on $[a,\infty[$, where $a>0$, but not uniformly continuous on $]0,\infty[$.

Example: The function $f(x)=\sin x^2$ is not uniformly continuous on $[0,\infty[$.

Example: The function $f(x)=\sqrt{x}$ is uniformly continuous in $[0,1]$.

Example: The function $f(x)=\sin\dfrac{1}{x}$ is not uniformly continuous on $]0,\infty[$.

Example: The function $f(x)=1/x$ is not uniformly continuous on $]0,1[$, but it is uniformly continuous on $[a,\infty[$, where $a>0$.

6.5.1. Non-Uniform Continuity Criterion

Let $I \subseteq \mathbb{R}$ and let $f : I \to \mathbb{R}$. Then the following statements are equivalent:

(i) f is not uniformly continuous on I.

(ii) There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ, u_δ in I such that $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$.

(iii) There exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in I such that $\lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

Application: We can apply this result to claim that $f(x) = 1/x$ is not uniformly continuous on $I = (0, \infty)$. For, if $x_n = 1/n$ and $u_n = 1/(n+1)$, then we have $\lim(x_n - u_n) = 0$, but $|g(x_n) - g(u_n)| = 1$ for all $n \in \mathbb{N}$.

Result: A function which is uniformly continuous on an interval is continuous on that interval.

***Note:* Converse of the result need not be true.**

Example: Let $f : (0,1) \to \mathbb{R}$ be a function defined as $f(x) = \dfrac{1}{x}$. Then f is continuous but not uniformly continuous on $(0,1)$.

6.5.2. Uniform Continuity Theorem

Let I be a closed and bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

Alternative solution: The function $f(x) = x/(x+1)$ is uniformly continuous for $x \in [0,2]$

Here $f(x)$ is a rational function and so it is continuous for every real number other than zeros of the denominator of polynomial function $x+1$. But $x+1=0 \Rightarrow x=-1$, which does not belong to $[0,2]$. Hence $f(x)$ is continuous on the closed interval $[0,2]$.

Since every function which is continuous in a closed interval is also uniformly continuous in that interval, it follows that $f(x) = x/(x+1)$ is uniformly continuous in $[0,2]$.

6.5.3. Lipschitz Function

Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function if there exists a constant $K > 0$ such that $|f(x) - f(u)| \leq K|x-u|$ for all $x, u \in A$. Then f is said to be a **Lipschitz function** (or to satisfy a **Lipschitz condition**) on A.

Result: If $f : I \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on I.

Remark:

Converse of the above result need not be true. *i.e.,* A uniformly continuous function on the interval I may not be Lipschitz function on I.

Example: Let $g(x) = \sqrt{x}$ for x in the closed and bounded interval $I = [0,2]$. Since g is continuous on I. Then g is uniformly continuous on I. However, g is not a Lipschitz function on I.

6.5.4. Continuous Extension Theorem

Let $I=(a,b)$ be an interval. Then, $f:I\to\mathbb{R}$ is uniformly continuous on I if and only if it can be defined at the point a and b such that the extended function is continuous on $[a,b]$

Application: The function defined by $f(x)=x\sin\left(\frac{1}{x}\right)$ is uniformly continuous on $(0,1)$

Reason: Since the extension function $g:[0,1]\to\mathbb{R}$ defined by

$$g(x)=\begin{cases} x\sin\left(\frac{1}{x}\right) & x\neq 0 \\ 0 & x=0 \end{cases} \text{ is continuous on } [0,1]$$

6.5.5. Some Important Sufficient Conditions For Uniform Continuity

1. If f is continuous on $\mathbb{R}$ then it is uniformly continuous on every interval of finite length. But not conversely.

   Example: A function $f(x)=x^2$ is uniformly continuous on every interval of finite length but not uniformly continuous on $\mathbb{R}$.

2. If f is differentiable on (a,b) and derivative of f is bounded. Then f is uniformly continuous on (a,b) but not conversely (we will prove this by LMVT).

   Example: A function $f:(0,1)\to\mathbb{R}$ defined as $f(x)=\sqrt{x}$ is uniformly continuous on $(0,1)$ but derivative is not bounded.

3. If f is periodic on $\mathbb{R}$ and continuous on $\mathbb{R}$. Then it is uniformly continuous on $\mathbb{R}$.
4. Graph of a function is parallel to x-axis then f must be uniformly continuous on its given domain.
5. Graph of f is parallel to y-axis (or fluctuate) then f cannot be uniformly continuous on its given domain.
6. If f is continuous on $[a,\infty)$, $(a\in\mathbb{R})$ and limit at ∞ exist finitely. Then f is uniformly continuous on $[a,\infty)$.
7. If f is uniformly continuous on (a,b) & (b,c) and continuous at $x=b$. Then f is uniformly continuous on (a,c)

Result: Let $f:I\to\mathbb{R}$ be a uniformly continuous function on I and $\langle x_n\rangle$ be a Cauchy sequence in I. Then $\langle f(x_n)\rangle$ is also Cauchy Sequence.

Remark:

(a) This result tells us that image of a Cauchy Sequence under uniformly continuous map is Cauchy sequence.

(b) Uniform continuity $\not\Rightarrow$ boundedness i.e., A uniformly continuous function may not be bounded.

Example: A function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x$.

(c) Continuity + boundedness $\not\Rightarrow$ Uniform continuity i.e., A continuous bounded function may not be uniform continuous.

Example: A function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\sin\left(x^2\right)$.

6.5.6. Algebra of Uniformly Continuous Functions

Let f and g are uniformly continuous functions on the interval I. Then

(i) $\alpha f + \beta g$ is uniformly continuous on I, $\forall \alpha, \beta \in \mathbb{R}$

(ii) If f and g both are bounded functions. Then $f.g$ is uniformly continuous function.

(iii) If $f(x) \geq k > 0 \ \forall x \in I$ where $k \in \mathbb{R}$. Then $\frac{1}{f}$ is uniformly continuous on I.

Remark:

(a) Condition of boundedness of f and g cannot be relaxed in (ii)

Example: Let us consider functions $f, g : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = g(x) = x \ \forall x \in \mathbb{R}$.

Then both f and g are uniformly continuous on $\mathbb{R}$ but $(f.g)(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

(b) The condition $f(x) \geq k > 0$ cannot be relaxed in (iii)

Example: Let us consider a functions $f : (0,1) \to \mathbb{R}$ defined as $f(x) = x \ \forall x \in (0,1)$. Then f is uniformly continuous. But $\frac{1}{f} : (0,1) \to \mathbb{R}$ which is defined as $\left(\frac{1}{f}\right)(x) = \frac{1}{x}$ is not uniformly continuous

CHAPTER 7

DIFFERENTIABILITY

7.1. Definition

Let f be a real valued function defined on an interval $[a,b]$ i.e., $f:[a,b]\to\mathbb{R}$. Let $a<c<b$.

(i) **Left Hand Derivative:** The left hand derivative of a function f at $x=c$, denoted by $Lf'(c)$ or $f'(c-0)$ is defined as $Lf'(c)=\lim\limits_{x\to c^-}\dfrac{f(x)-f(c)}{x-c}$, provided the limit exists.

(ii) **Right Hand Derivative:** The right hand derivative of a function f at $x=c$, denoted by $Rf'(c)$ or $f'(c+0)$, is defined as $Rf'(c)=\lim\limits_{x\to c^+}\dfrac{f(x)-f(c)}{x-c}$, provided the limit exists.

(iii) If $Lf'(c)=Rf'(c)$, we say that the function f is derivable at $x=c$. The common limit is called the derivative of f at $x=c$ and is denoted by $f'(c)$. Thus $f'(c)=\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}$.

(iv) If $Lf'(c)\neq Rf'(c)$, we say that the function f is not derivable at $x=c$.

Example:

(a) Let us define $f:\mathbb{R}\to\mathbb{R}$ as $f(x)=\begin{cases}x^2 & x\in\mathbb{Q}\\ 2x-1 & x\in\mathbb{Q}^c\end{cases}$. Then f is differentiable at $x=1$.

(b) Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined as $f(x)=\begin{cases}x^{\alpha}\sin\left(\dfrac{1}{x^{\beta}}\right), & \alpha,\beta>0,\ x\neq 0\\ 0 & x=0\end{cases}$. Then f is differentiable at $x=0\Leftrightarrow\alpha>1$.

(c) Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined as $f(x)=|x|+|x-1|\ \forall x\in\mathbb{R}$. Then f is not differentiable at $x=0,1$.

(d) The function $f(x)=x|x|$ is derivable at the origin.

Reason: We have

$$Lf'(0)=\lim_{x\to 0-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x(-x)}{x}=0,$$

$$Rf'(0)=\lim_{x\to 0+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x(x)}{x}=0.$$

$\therefore\ Lf'(0)=Rf'(0)=0$.

Hence $f'(0)=0$

(e) The function f defined by $f(x)=\begin{cases} x^2+3x+a, & \text{if } x\le 1 \\ bx+2, & \text{if } x>1 \end{cases}$ is given to be derivable for every x. Find a and b.

Solution: Since f is derivable for every x, f must be derivable at $x=1$ and hence, f must be continuous at $x=1$. Let $h>0$. Then f is continuous at $x=1$.

$$\Rightarrow f(1)=f(1+0)=f(1-0)$$

$$\Rightarrow 4+a=\lim_{h\to 0} f(1+h)=\lim_{h\to 0} f(1-h)$$

$$\Rightarrow 4+a=\lim_{h\to 0}\{b(1+h)+2\}=\lim_{h\to 0}\{(1-h)^2+3(1-h)+a\}$$

$$\Rightarrow 4+a=b+2=4+a \;\Rightarrow\; a-b=-2 \qquad \ldots(1)$$

Again, f is differentiable at $x=1 \;\Rightarrow\; Rf'(1)=Lf'(1)$

$$\Rightarrow \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$\Rightarrow \lim_{h\to 0}\frac{b(1+h)+2-(4+a)}{h}=\lim_{h\to 0}\frac{(1-h)^2+3(1-h)+a-(4+a)}{-h}$$

$$\Rightarrow \lim_{h\to 0}\frac{b-a+bh-2}{h}=\lim_{h\to 0}\frac{h^2-5h}{-h}$$

$$\Rightarrow \lim_{h\to 0}\frac{b-(b-2)+bh-2}{h}=\lim_{h\to 0}(5-h), \text{ using (1).}$$

$$\Rightarrow b=5$$

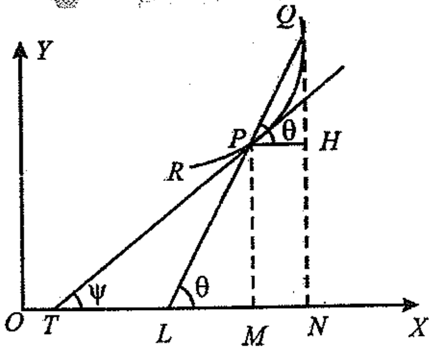
Since $b=5$, so (1) gives $a=b-2=5-2=3$. Thus, $a=3, b=5$.

7.2. Geometrical Meaning of the Derivative

Let $f:[a,b]\to\mathbb{R}$ be derivable at $c\in]a,b[$. Let RQ be the graphical representation of the function $y=f(x)$. We take two neighboring points $P[c,f(c)]$ and $Q[c+h,f(c+h)]$. On the curve such that $a<c+h<b$. Join PQ and produce it to meet the x-axis at L. Let QL make an angle θ with x-axis. Draw the tangent PT at P which makes an angle ψ with x-axis. Draw PM and QN perpendiculars from P and Q on the x-axis. Also draw PH perpendicular from P on QN. Then, we have

$QH=QN-HN=QN-PM=f(c+h)-f(c)$ and

$PH=MN=ON-OM=(c+h)-c=h$.



$\therefore$ Slope of $PQ = \tan\theta = \dfrac{QH}{PH} = \dfrac{f(c+h)-f(c)}{h}$

From the figure, we see that as $h \to 0$, the point Q approaches the point P along the graph of f and, hence, the chord PQ approaches the tangent line PT as its limiting position. Thus, as $h \to 0, \theta \to \psi$.

$$\therefore \tan\psi = \lim_{Q \to P} \tan\theta = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = f'(c),$$

Showing that $f'(c)$ is the slope of the tangent to the curve $y = f(x)$ at the point $(c, f(c))$.

Note: If there is a sharp edge in the curve then at that point f is not differentiable.

Remark:

1. Continuity is a necessary but not a sufficient condition for the existence a finite derivative.

i.e., every differentiable function is continuous but converse need not be true.

Proof: Let f be derivable at the point c so that $\lim_{x \to c} \dfrac{f(x)-f(c)}{x-c}$ exists.

We have $f(x)-f(c) = \left[\dfrac{f(x)-f(c)}{x-c}\right](x-c); x \neq c$

$$\Rightarrow \quad \lim_{x \to c}\left[f(x)-f(c)\right] = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \lim_{x \to c}(x-c)$$

$$\Rightarrow \quad \lim_{x \to c} f(x)-f(c) = f'(c).0 = 0$$

$$\Rightarrow \quad \lim_{x \to c} f(x) = f(c)$$

$\Rightarrow \quad f$ is continuous at c.

Thus, continuity is a necessary condition for differentiability but it is not a sufficient condition for the existence of a finite derivative as shown in the following example. Thus, the converse may not be true.

Example: Let f be defined by $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$.

Then f is continuous at 0, but it is not derivable at 0.

2. If at $x = c, f$ has Left Hand Derivative and Right Hand Derivative both (may or may not be equal)

Then f is continuous at $x = c$.

3. If $f:[a,b] \to \mathbb{R}$ such that f is differentiable on (a,b) and Right Hand Limit of $\dfrac{f(x)-f(b)}{x-b}$ exist finitely at $x = b$ and Left Hand Limit of $\dfrac{f(x)-f(a)}{x-a}$ exist finitely at $x = a$. Then f is differentiable on $[a,b]$

7.3. Algebra of Differentiability

Let f and g are two differentiable functions at $c \in S \subseteq \mathbb{R}$. Then

(i) $\alpha f + \beta g$ is differentiable at c $\forall\, \alpha, \beta \in \mathbb{R}$ and $(\alpha f + \beta g)'(c) = \alpha f'(c) + \beta g'(c)$

(ii) $(f.g)$ is differentiable at c and $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$

(iii) If $g(c) \neq 0$. Then $\frac{f}{g}$ is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{(g(c))^2}$$

Remark:

(a) Converse of the above result need not be True.

Example: Let $f : \mathbb{R} \to \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}, g(x) = \begin{cases} -1 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

Clearly, f and g are not differentiable at $x = 0$. But $f + g, f.g, \frac{f}{g}$ all are differentiable at $x = 0$.

(b) Above results can be extended for finite numbers of differentiable functions (By induction).

(c) If f is differentiable on $S \subseteq \mathbb{R}$ and g is not differentiable on S. Then $f \pm g$ cannot be differentiable on S. Also if $f(x) \neq 0$ $\forall\, x \in S$, then $f.g$ is not differentiable on S.

Result: Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. If c is an interior point of $[a,b]$ and $f'(c) > 0$ then $\exists\, \delta > 0$ such that $f(x) > f(c)$ $\forall\, x \in [c, c+\delta]$, $f(x) < f(c)$ $\forall\, x \in [c-\delta, \delta]$.

Similarly if $f'(c) < 0$ then $\exists\, \delta > 0$ such that $f(x) < f(c)$ $\forall\, x \in [c, c+\delta]$

Also, it may be shown that

(a) $f'(a) > 0 \Rightarrow f(x) > f(a)$ $\forall\, x \in [a, a+\delta]$

(b) $f'(a) < 0 \Rightarrow f(x) < f(a)$ $\forall\, x \in [a, a+\delta]$

(c) $f'(b) > 0 \Rightarrow f(x) < f(b)$ $\forall\, x \in [b-\delta, b]$

(d) $f'(b) < 0 \Rightarrow f(x) > f(b)$ $\forall\, x \in [b-\delta, b]$

7.3.1 Darboux's Theorem

If a function is derivable in a closed interval $[a,b]$ and $f'(a), f'(b)$ are of opposite signs, then there exist at least one point c in the open interval (a,b) such that $f'(c) = 0$.

Note : If g is derivative of any function f then g satisfy IVP.

Remark:

1. If f is derivable in $[a,b]$ and $f'(a) \neq f'(b)$, then for each number k lying between $f'(a)$ and $f'(b)$, $\exists$ some point $c \in (a,b)$ such that $f'(c) = k$.

   Reason: We are given that $f'(a) < k < f'(b)$ ...(1)

   Let $g(x) = f(x) - kx \quad \forall\, x \in [a,b]$. ...(2)

   Since, f is derivable in $[a,b]$ and kx is also derivable in $[a,b]$, therefore, by (2), g is derivable in $[a,b]$.

   Now $g'(x) = f'(x) - k$ ...(3)

   $\Rightarrow \; f'(c) - k = 0$, by (3)

   Hence, $f'(c) = k$

2. If f be defined and derivable on $[a,b]$, $f(a) = f(b) = 0$ and $f'(a)$ and $f'(b)$ are of the same sign, then f must vanish at least once in (a,b).

   Reason: We are given that $f(a) = f(b) = 0$.

   Since $f'(a)$ and $f'(b)$ are of the same sign, we may take

   $f'(a) > 0$ and $f'(b) > 0$.

   Now $f'(a) > 0 \Rightarrow \exists$ some $\delta_1 > 0$ such that

   $f(x) > f(a) = 0 \quad \forall\, x \in \,]a, a+\delta_1]$.

   Again $f'(b) > 0 \Rightarrow \exists$ some $\delta_2 > 0$ such that

   $f(x) > f(b) = 0 \quad \forall\, x \in [b-\delta_2, b[$.

   Thus $f(a+\delta_1) > 0$ and $f(b-\delta_2) < 0$. ...(1)

   Obviously, $[a+\delta_1, b-\delta_2] \subset [a,b]$ ...(2)

   Since f is derivable in $[a,b]$, f is continuous in $[a,b]$ and consequently by (2), f is continuous in $[a+\delta_1, b-\delta_2]$, where $f(a+\delta_1)$ and $f(b-\delta_2)$ are of opposite signs, by (1). Thus, there exists some $c \in \,]a+\delta_1, b-\delta_2[\, \subset \,]a,b[$ such that $f(c) = 0$. Hence f must vanish at least once in $]a,b[$.

3. If f is derivable on $[a,b]$, $f(a) = f(b) = 0$ and $f(x) \neq 0$ for any x in (a,b) then $f'(a)$ and $f'(b)$ must have opposite signs.

   Reason: Let, if possible, $f'(a)$ and $f'(b)$ be of the same sign. Also $f(a) = f(b) = 0$. Then, there exists some $c \in \,]a,b[$ such that $f(c) = 0$.

This contradicts the given hypothesis that $f(x) \neq 0$ for all x in $]a,b[$. Hence $f'(a)$ and $f'(b)$ must be of opposite signs.

4. If f is derivable at a point c, then $|f|$ is also derivable at c, provided $f(c) \neq 0$. However result may not hold when $f(c) = 0$.

5. Let $f:[-1,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0, -1 \le x < 0 \\ 1, \quad 0 < x \le 1 \end{cases}$

   Then there does not exist a function g such that $g'(x) = f(x)\ \forall x \in [-1,1]$.

   Reason: Let, if possible, there exist a function $g:[-1,1] \to \mathbb{R}$ such that $g'(x) = f(x) = \begin{cases} 0, -1 \le x \le 0 \\ 1, \quad 0 < x \le 1. \end{cases}$

   It follows that g is derivable on $[-1,1]$ and $g'(-1) \neq g'(1)$ $[\because g'(-1) = 0 \text{ and } g'(1) = 1]$.

   $\Rightarrow$ g' must assume every value between 0 and 1. $[\because g'(-1) = 0 \text{ and } g'(1) = 1]$

   But this is impossible, since $g'(x) = 1\ \forall x \in]0,1[$.

   Hence there does not exist any function g such that $g'(x) = f(x)\ \forall x \in [-1,1]$.

7.4. Local Maxima & Local Minima

Let $f: I \to \mathbb{R}$ and c be an interior point of the interval I. Then

1. $f(c)$ is said to be a local maximum value of the function f, if there exist some neighbourhood $(c-\delta, c+\delta)$ of c, such that $f(c) > f(x)\ \forall x \in (c-\delta, c+\delta)$ other than c and c is called point of local maxima.
2. $f(c)$ is said to be a minimum value of the function f, if there exists some neighbourhood $]c-\delta, c+\delta[$ of c, such that $f(c) < f(x)\ \forall x \in]c-\delta, c+\delta[$ other than c and c is called point of local minima.
3. $f(c)$ is said to be an extreme value of f, if it is either a maximum or a minimum value.
4. **Point of inflexion:** An inflection point, point of inflection is a point on a curve at which the curve changes from being concave (concave downward) to convex (concave upward), or vice versa.

   Example: Let $f(x) = x^3$. Then $(0,0)$ is point of inflexion of f.

Note: For $f(c)$ to be an extreme value, $f(c) - f(x)$ must keep the same sign for every point x other than c in some neighbourhood of c.

Interior Extremum Theorem: Let c be an interior point of the interval I at which $f: I \to \mathbb{R}$ has an extremum. If the derivative of f at c exist then $f'(c)=0$

Remark:

(a) This theorem is applicable only on interior points of the domain.

Reason: Let $f:[0,1]\to\mathbb{R}$ be defined as $f(x)=x$. Then at $x=0$, f has minimum value 1 and at $x=1$, f has maximum value but neither point is a zero of the derivative of f.

(b) Converse of the theorem need not be true i.e., if $f'(c)=0$, then f may have neither a maximum value nor a minimum value.

Example: Let $f:[0,1]\to\mathbb{R}$ defined as $f(x)=x^3$. Such that $f'(x)=3x^2$

Then $f'(0)=0$

But f has neither a maximum value nor a minimum value at $x=0$.

(c) A function may have maximum or a minimum value at a point without being derivable at that point.

Example: Let us define a function $f(x)=|x|$.

Then f is not derivable at $x=0$ but f has the minimum at $x=0$.

(d) Let $f: I\to\mathbb{R}$ be continuous on an interval I and suppose that f has a relative extremum at an interior point c of I. Then either the derivative of f at c does not exist, or it is equal to zero.

Results:

1. If a function f is continuous in $[a,b]$ such that

(i) $f'(x)>0\ \forall\, x\in\,]a,b[$, then f is increasing in $[a,b]$,

(ii) $f'(x)<0\ \forall\, x\in\,]a,b[$, then f is decreasing in $[a,b]$. Further if f is defined in $[a,b]$ and $a<c<b$, then

(iii) $f'(c)>0 \Rightarrow f$ is increasing in $]c-\delta_1, c+\delta_1[$ for some $\delta_1>0$.

(iv) $f'(c)<0 \Rightarrow f$ is decreasing in $]c-\delta_2, c+\delta_2[$ for some $\delta_2>0$.

2. First derivative test for extreme values

Let a function f be derivable in a neighbourhood of c, where f has an extreme value at c. Then $f(c)$ is a maximum value if the sign of f' changes from plus to minus and $f(c)$ is a minimum value if the sign of f' changes from minus to plus as x passes through c.

Applications:

Example: Examine the function x^3-6x^2+9x-4 for maximum and minimum values.

Solution: Let $f(x)=x^3-6x^2+9x-4$. Then

$$f'(x)=3x^2-12x+9=3\left(x^2-4x+3\right)$$

or $f'(x)=3(x-1)(x-3)$

$\therefore\ f'(x)=0$ for $x=1,3$.

Now for $x<1$, $f'(x)=(-).(-)>0$,

and for $x>1$, $f'(x)=(+).(-)<0$.

Thus $f'(x)$ changes sign from plus to minus in going through $x=1$. Hence the function has a maximum at $x=1$ and $f(1)=0$ is a maximum value of f.

Now for $x<3, f'(x)=(+).(-)<0$,

and for $x>3, f'(x)=(+).(+)>0$.

Thus $f'(x)$ changes sign from minus to plus in going through $x=3$. Hence the function has minimum at $x=3$ and $f(3)=-4$ is a minimum value of f.

3. Second derivative test for extreme values

Let f be derivable on an interval $]c-\varepsilon, c+\varepsilon[$ and let $f'(c)=0$.

(i) If $f'(c)=0$ and $f''(c)<0$, then f has a maximum value at $x=c$.

(ii) If $f'(c)=0$ and $f''(c)>0$, then f has a minimum value at $x=c$.

Example: Find the maximum and minimum values of the function $f(x)=12x^5-45x^4+40x^3+6, x\in\mathbb{R}$

Solution: We have

$$f'(x)=60x^4-180x^3+120x^2$$

$$=60x^2\left(x^2-3x+2\right)=60x^2(x-1)(x-2).$$

The extreme values of the function f are given by $f'(x)=0$

i.e., $x=0,1,2$.

Now $f''(x)=240x^3-540x^2+240x$

or $f''(x)=60\left(4x^3-9x^2+4x\right)$.

$\therefore\ f''(2)=240>0\Rightarrow f$ has a minimum at $x=2$.

and $f''(1)=-60<0\Rightarrow f$ has a maximum at $x=1$.

Since $f''(0)=0$, we find out the third derivative. We have

$$f'''(x)=720x^2-1080x+240\Rightarrow f'''(0)=240\neq 0.$$

So f has neither a maximum nor a minimum at $x=0$.

Minimum value $= f(2) = -10$.

Maximum value $= f(1) = 12 - 45 + 40 + 6 = 13$.

Remark:

(a) Let $f(x) = x^3 \ \forall x \in \mathbb{R}$. Then $f'(0) = 0$ and $f''(0) = 0$, but the function f has neither a maximum nor a minimum at $x = 0$.

(b) Let $f(x) = x^4 \ \forall x \in \mathbb{R}$. Then $f'(0) = 0$ and $f''(0) = 0$. Clearly, the curve $y = x^4$ is symmetrical about the y-axis, passes through the origin and lies above the x-axis only. Hence the function f has a minimum at $x = 0$.

(c) Let $f(x) = -x^4 \ \forall x \in \mathbb{R}$. Then $f'(0) = 0, f''(0) = 0$, and f has a maximum at $x = 0$.

4. Let f be a function defined on an interval $[a,b]$ and $a < c < b$. Let $f'(c) = f''(c) = \ldots = f^{n-1}(c) = 0$ and $f^n(c) \neq 0$.

(i) If n is odd, then f has neither a maximum nor a minimum at $x = c$.

(ii) If n is even, then f has a maximum at $x = c$ if $f^n(c) < 0$, and f has a minimum at $x = c$ if $f^n(c) > 0$.

7.5. Mean Value Theorems

1. Rolle's Theorem:

Let f be a function defined on $[a,b]$ such that

(i) f is continuous on $[a,b]$

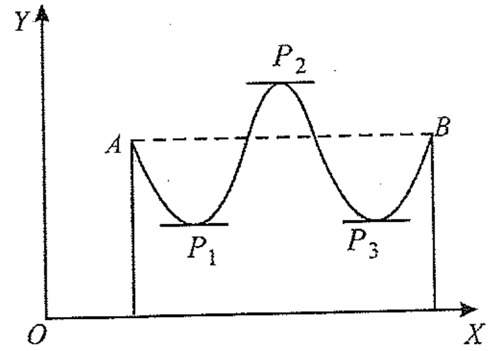
(ii) f is differentiable on (a,b)

(iii) $f(a) = f(b)$

Then $\exists \ c \in (a,b)$ such that $f'(c) = 0$.

Geometrical interpretation of Rolle's Theorem:

Let A and B be the points on the graph $AP_1P_2P_3B$ of the function $y = f(x)$ corresponding to $x = a$ and $x = b$ respectively. Then, geometrically, Rolle's theorem asserts that there is at least one point between $x = a$ and $x = b$, at which the tangent to the curve of the function, is parallel to x-axis. In the figure, we have shown the possibility of three points P_1, P_2 and P_3 where the tangent is parallel to x-axis.



Remark:

(a) This theorem is useful to produce the zero of derivative of the function.

(b) A non constant function satisfy the conditions of Rolle's theorem cannot be monotonic.

(c) This theorem tells us that between two distinct zeros of a differentiable function f there lies a zero of f'

(d) Third condition of Rolle's theorem cannot be relaxed.

Example: Let $f:[2,3]\to\mathbb{R}$ be defined as $f(x)=x$.

Then f is continuous on $[2,3]$ and differentiable on $(2,3)$ but $\nexists\, c\in(2,3)$ such that $f'(c)=0$.

Infact Rolle's theorem fails to hold good for a function which does not satisfy any condition of the theorem.

(e) This theorem only gives the existence of a root of a derivative of the function. i.e., it may be possible that derivative of function has finitely many or infinite zeroes.

(f) The condition of Rolle's theorem are only sufficient but not necessary for $f'(x)$ to vanish at some point in $[a,b]$

Example: Consider the function $f(x)=\begin{cases}0 & \text{when } 0\le x\le 1\\ x+1 & \text{when } 1<x\le 2\end{cases}$. We can easily show that $f(x)$ is discontinuous and not derivable at $x=1$. Thus $f(x)$ is not continuous on $[0,2]$ and $f(x)$ is not derivable on $(0,2)$. Also $f(0)\ne f(2)$ but $f'(x)=0$ $\forall\, x\in[0,1]\subset[0,2]$

2. Lagrange's Mean Value Theorem (LMVT)

Let f be a function defined on $[a,b]$ such that

(i) f is continuous on $[a,b]$

(ii) f is differentiable on (a,b)

Then $\exists\; c\in(a,b)$ such that $\dfrac{f(b)-f(a)}{b-a}=f'(c)$

Remark:

(a) The c obtained in LMVT need not to be unique i.e., it may be finite or countably infinite or uncountable

(b) The collection of such c is non empty.

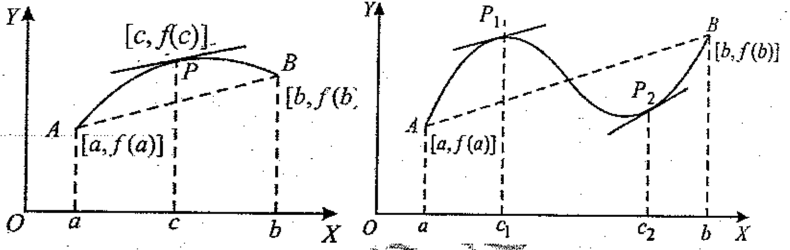
(c) LMVT is obtained by Rolle's theorem.

Geometrical Interpretation of Lagrange's Mean Value Theorem:

Let A and B be the points on the graph of the function $y=f(x)$ corresponding to $x=a$ and $x=b$ respectively. Then the coordinates of A and B are $(a,f(a))$ and $(b,f(b))$ respectively.

Now, the slope of chord $AB=[f(b)-f(a)]/(b-a)$.

Since $f(x)$ is continuous on $[a,b]$ and differentiable on $]a,b[$, figure shows that there is at least one point between A and B the tangent at which is parallel to the chord AB.



If 'c' be the abscissa of P, then slope of tangent at P is $f'(c)$. Since chord AB is parallel to the tangent at P, we have $[f(b)-f(a)]/(b-a)=f'(c)$.

Thus, interpreted geometrically, Lagrange's mean value theorem says that the tangent to the graph $y=f(x)$ at some suitable point between a and b is parallel to the chord joining the points on the graph with abscissa a and b.

3. Generalized form of Lagrange's Mean Value Theorem:

If a function f defined in $[a,a+h]$ is such that

(i) f is continuous in $[a,a+h]$

(ii) f is derivable in $]a,a+h[$

Then there exists some $\theta\in]0,1[$ such that $f(a+h)-f(a)=hf'(a+\theta h)$

Some useful deduction from the Mean Value Theorem:

(i) If f is defined and continuous on $[a,b]$ and is derivable on $]a,b[$, and if $f'(x)=0$ for all x in $]a,b[$, then $f(x)$ has a constant value throughout $[a,b]$.

(ii) If $f(x)$ and $g(x)$ are both defined and continuous on $[a,b]$, and are derivable on $]a,b[$, and if $f'(x)=g'(x)\ \forall x\in]a,b[$, then $f(x)$ and $g(x)$ differ by a constant on $[a,b]$.

(iii) If f is continuous on $[a,b]$, and $f'(x)\geq 0$ in $]a,b[$, then f is increasing in $[a,b]$.

(iv) If f is continuous on $[a,b]$ and $f'(x)>0$ in $]a,b[$, then f is strictly increasing in $[a,b]$.

(v) If f is continuous on $[a,b]$ and $f'(x)\leq 0$ in $]a,b[$, then f is decreasing in $[a,b]$.

(vi) If f is continuous on $[a,b]$ and $f'(x)<0$ in $]a,b[$, then f is strictly decreasing in $[a,b]$.

Example: Let a function f be defined on $\mathbb{R}$ such that $|f(x)-f(y)|<(x-y)^2$ for all x and y. Then f must be constant function

Solution: Given $|f(x)-f(y)|<(x-y)^2 \quad \forall x,y\in\mathbb{R}$ ...(1)

Let c be any real number. Then, for $x\neq c$, (1) gives

$$\left|\frac{f(x)-f(c)}{x-c}\right|\leq|x-c| \qquad ...(2)$$

Let $\varepsilon>0$ be given. Then, we choose $\delta=\varepsilon$ and so (2) gives

$$\left|\frac{f(x)-f(y)}{x-c}-0\right|<\varepsilon, \text{ whenever } |x-c|<\delta$$

$$\Rightarrow \lim_{x\to c}\frac{f(x)-f(c)}{x-c}=0\Rightarrow f'(c)=0 \qquad ...(3)$$

Since c is any point on $\mathbb{R}$, (3) shows that $f'(x)=0 \quad \forall x\in\mathbb{R}$.

Hence, f must be constant function.

4. **Cauchy Mean Value Theorem:**

Let f and g be two functions defined on $[a,b]$ such that

(i) f and g are continuous in $[a,b]$

(ii) f and g are derivable in (a,b)

(iii) $g'(x)\neq 0$ for each $x\in(a,b)$ and $g(a)\neq g(b)$

Then there exist at least one point $c\in(a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

Put Your Own Notes

7.6. Higher Order Derivatives

We know that the existence of the derivative f' of a function f at a point c implies the existence and continuity of the function in a neighbourhood of c. The derivative of the function f' at c (in case it exists), is called the second derivative of f at c and denoted by $f''(c)$. Evidently, the existence of $f''(c)$ implies the existence and continuity of f' in a neighbourhood of c.

Higher order derivatives can be similarly defined. The derivative of f^{n-1} at c (in case it exists) is called the nth derivative of f at c and is denoted by $f^{n}(c)$.

1. **Taylor's Theorem with Lagrange's Remainder**

   If a function f defined on $[a,a+h]$ is such that

   (i) The $(n-1)th$ derivative f^{n-1} is continuous in $[a,a+h]$,

   (ii) The $(n-1)th$ derivative f^{n-1} is derivable in $]a,a+h[$,

   then there exists some $\theta, 0<\theta<1$, such that

   $$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+R_n$$ where

   $$R_n=\frac{h^n}{n!}f^n(a+\theta h).$$

   [Lagrange's Remainder]

2. **Taylor's Theorem with Cauchy's Remainder**

   If a function f defined on $[a,a+h]$ is such that

   (i) The $(n-1)th$ derivative f^{n-1} is continuous in $[a,a+h]$,

   (ii) The $(n-1)th$ derivative f^{n-1} is derivable in $]a,a+h[$,

   then there exists some $\theta, 0<\theta<1$ such that

   $$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+R_n \quad ,$$ Where

   $$R_n=\frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h).$$

   [Cauchy's Remainder]

3. **Maclaurin's Theorem (With Lagrange's Reminder)**

   On taking $a=0$ in Taylor Theorem, we get **Maclaurin's Theorem,** which may be stated as follows:

   If a function f defined on $[0,h]$ is such that

   (i) f^{n-1} is continuous in $[0,h]$,

(ii) f^{n-1} is derivable in $]0,h[$,

then for each x in $[0,h]$, there exists a real number θ between 0 and 1 such that

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+...+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^n(\theta x)$$

(Lagrange's remainder)

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+...+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)$$

$$+\frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^n(\theta x)$$

(Cauchy's remainder)

7.7. Indeterminate Forms

We shall now discuss the evaluation of limits of functions generally known as Indeterminate forms. They are not indeterminate but have acquired this name by usage of the word.

In general, the limit of $\frac{\phi(x)}{\psi(x)}$ when $x \to a$, in case the limits of both the functions exist, is equal to the limit of the numerator divided by the limit of the denominator. But what happens when both these limits are zero? The division $\frac{0}{0}$ then becomes meaningless. A case like is known as Indeterminate form. Other such forms are $\frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, 1^\infty$, and ∞^0. Ordinary methods of evaluating the limits are of little help. Particular methods are required to evaluate these limits. We shall now discuss these particular methods, generally called L'Hospital rule, due to the French mathematician L'Hospital.

It should, however, be clearly understood, that in what follows, we do not find the value of $\frac{0}{0}$ or of any of the other indeterminate forms. We only find the limits of combinations of functions which assume these forms when the limits of functions are taken separately.

L'Hopital's Rule for $\frac{0}{0}$ Form

If f and g be two functions such that

(i) $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0$,

(ii) $f'(x), g'(x)$ exist and $g(x)\neq 0, g'(x)\neq 0\ \forall x\in]a-\delta,a+\delta[, \delta>0$, except possibly at a,

(iii) $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists,

Then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Note: If $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ does not exist then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ may exist.

For example: $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x-\cos x}=1$ while $\lim\limits_{x\to\infty}\dfrac{1+\cos x}{1+\sin x}$ does not exist.

Remark: It may happen that $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ is $\dfrac{0}{0}$ form. In this case, we can extend the L'Hopital's rule as follows:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)},$$

where f' and g' satisfy the conditions of L'Hopital's rule.

This rule can be generalized as follows:

If $f(x)$ and $g(x)$ are functions such that $\lim\limits_{x\to a}\dfrac{f^{(n-1)}(x)}{g^{(n-1)}(x)}$ represents the indeterminate form $\dfrac{0}{0}$ and the functions $f^{(n)}(x)$ and $g^{(n)}(x)$ satisfy the conditions of L'Hopital's rule, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\ldots=\lim_{x\to a}\frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

Examples:

1. $\lim\limits_{x\to 0}\dfrac{1-\cos x^2}{x^2\sin x^2}=\dfrac{1}{2}.$

Solution: We have $\dfrac{1-\cos x^2}{x^2\sin x^2}=\dfrac{1-\cos x^2}{x^4}\cdot\dfrac{x^2}{\sin x^2}.$

Since $\lim\limits_{x\to 0}\dfrac{x^2}{\sin x^2}=1$, $\therefore \lim\limits_{x\to 0}\dfrac{1-\cos x^2}{x^2\sin x^2}=\lim\limits_{x\to 0}\dfrac{1-\cos x^2}{x^4}\left(\dfrac{0}{0}\right)$

$$=\lim_{x\to 0}\frac{2x\sin x^2}{4x^3} \qquad \textbf{(by L'Hopital's rule)}$$

$$=\lim_{x\to 0}\frac{1}{2}\cdot\frac{\sin x^2}{x^2}=\frac{1}{2}.1=\frac{1}{2}.$$

2. $\lim\limits_{x\to 0}\dfrac{\tan x-x}{x^2\tan x}=\dfrac{1}{3}$

3. $\lim\limits_{x\to 0}\dfrac{a^x-1}{b^x-1}=\dfrac{\log a}{\log b}$, where $b\neq 1$

Solution: The given limit is $0/0$ form and so

$$\lim_{x\to 0}\frac{a^x-1}{b^x-1}=\lim_{x\to 0}\frac{a^x\log a}{b^x\log b}=\frac{\log a}{\log b}.$$

4. $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/x}-e+\frac{1}{2}ex}{x^2}=\dfrac{11e}{24}.$

Solution: Since $\lim_{x\to0}(1+x)^{1/x}=e$, the given limit is $0/0$ form.

Before we evaluate the limit, we find the expansion of $(1+x)^{1/x}$.

Let $y=(1+x)^{1/x}$, so that, $\log y=\frac{1}{x}\log(1+x)=\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...\right)$

$=\left(1-\frac{x}{2}+\frac{x^2}{3}-...\right)=1+\alpha$, where $\alpha=-\frac{x}{2}+\frac{x^2}{3}-.....$

Now $\log y=1+\alpha\Rightarrow y=e^{1+\alpha}=e.e^{\alpha}\Rightarrow y=e\left(1+\alpha+\frac{\alpha^2}{2!}+.....\right)$

or $y=e\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-...\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-...\right)+...\right]$.

$\therefore\ (1+x)^{1/x}=e\left[1-\frac{x}{2}+\frac{11}{24}x^2-...\right]$ Now $\lim_{x\to0}\frac{(1+x)^{1/x}-e+\frac{1}{2}ex}{x^2}$

$=\lim_{x\to0}\frac{e\left(1-\frac{x}{2}+\frac{11}{24}x^2-...\right)-e+\frac{1}{2}ex}{x^2}=\lim_{x\to0}\left(\frac{11e}{24}+\text{terms containing } x\right)=\frac{11e}{24}$.

Examples of the indeterminate forms of the type $\infty-\infty,\frac{\infty}{\infty},0.\infty$.

5. $\lim_{x\to0+}\frac{\cot x}{\log\sin x}$ does not exist.

Solution: The given limit is ∞/∞ form. By L'Hopital's rule for ∞/∞ form, we have

$=\lim_{x\to0+}\frac{\cot x}{\log\sin x}=\lim_{x\to0+}\frac{-\text{coses}^2 x}{\cot x}\left(\frac{\infty}{\infty}\right)$

$=\lim_{x\to0+}\frac{2\text{coses}^2\cot x}{-\text{coses}^2 x}=\lim_{x\to0+}(-2\cot x)=-\infty$.

6. $=\lim_{x\to0+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=0$.

Solution: $\lim_{x\to0+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$ is $\infty-\infty$ form. Such a form can be reduced to $0/0$ or ∞/∞ form. Now

$\lim_{x\to0+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim_{x\to0+}\frac{\sin x-x}{x\sin x}=\lim_{x\to0+}\frac{\sin x-x}{x^2}\left(\frac{x}{\sin x}\right)$

$=\lim_{x\to0+}\frac{\sin x-x}{x^2}\left(\frac{0}{0}\right)\quad\left(\because\lim_{x\to0+}\frac{x}{\sin x}=1\right)$

$=\lim_{x\to0+}\frac{\cos x-1}{2x}\left(\frac{0}{0}\right)=\lim_{x\to0+}\frac{-\sin x}{2}=0.$

7. $\lim\limits_{x\to\frac{1}{2}\pi}(\sec x-\tan x)=0$.

Solution: The given limit is $\infty-\infty$ form. We have

$$\lim_{x\to\frac{1}{2}\pi}(\sec x-\tan x)=\lim_{x\to\frac{1}{2}\pi}\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right)$$

$$=\lim_{x\to\frac{1}{2}\pi}\frac{1-\sin x}{\cos x}\left(\frac{0}{0}\right)=\lim_{x\to\frac{1}{2}\pi}\frac{-\cos x}{-\sin x}=\frac{0}{1}=0.$$

8. $\lim\limits_{x\to 1}\sec\left(\frac{\pi}{2}x\right)\log\left(\frac{1}{x}\right)=\frac{2}{\pi}$.

Solution: The given limit is $0.\infty$ form. We can write it as

$$\lim_{x\to 1}\frac{\log(1/x)}{\cos\left(\frac{\pi}{2}x\right)}\left(\frac{0}{0}\right)=\lim_{x\to 1}\frac{x\left(-\frac{1}{x^2}\right)}{-\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)}=\frac{2}{\pi}.$$

The Indeterminate Forms of the type $0^0, 1^\infty$ and ∞^0

To evaluate $\lim\limits_{x\to a}\left[f(x)\right]^{g(x)}$,

where $\lim\limits_{x\to a}f(x)=0,1$ or ∞ and $\lim\limits_{x\to a}g(x)=0,\infty$ or 0.

We write $y=\left[f(x)\right]^{g(x)}$

$\Rightarrow \log y=g(x)\log f(x)$

$\Rightarrow \lim\limits_{x\to a}\log y=\lim\limits_{x\to a}\left[g(x).\log f(x)\right]$.

The R.H.S. can be reduced to the form $0/0$ or ∞/∞, which can be evaluated. Suppose $\lim\limits_{x\to a}\left[g(x).\log f(x)\right]=l$.

Then $\lim\limits_{x\to a}\log y=l\Rightarrow\log\lim\limits_{x\to a}y=l\Rightarrow\lim\limits_{x\to a}y=e^l$.

Hence $\lim\limits_{x\to a}\left[f(x)\right]^{g(x)}=e^l$.

9. $\lim\limits_{x\to 0+}(\cot x)^{\sin x}=1$.

Solution: The given limit is ∞° form.

Let $y=(\cot x)^{\sin x}$, so that $\log y=\sin x\log\cot x$.

$$\therefore\ \lim_{x\to 0+}\log y=\lim_{x\to 0+}\sin x\log\cot x=\lim_{x\to 0+}\frac{\log\cot x}{\operatorname{cosec}x}\left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to 0+}\frac{-\operatorname{cosec}^2x(1/\cot x)}{-\cot x\operatorname{cosec}x},$$

$$=\lim_{x\to 0+}\frac{\operatorname{cosec}x}{\cot^2x}=\lim_{x\to 0+}\frac{\sin x}{\cos^2x}=0$$

$$\therefore\ \lim_{x\to 0+}\log y=0\Rightarrow\log\lim_{x\to 0+}y=0\Rightarrow\lim_{x\to 0+}y=e^0=1.$$

Hence $\lim\limits_{x\to 0+}(\cot x)^{\sin x}=1$.

10. $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{1/x^2}=e^{1/3}$.

Solution: The given limit is 1^∞ form.

Let $y=\left(\dfrac{\tan x}{x}\right)^{1/x^2}$, so that $\log y=\dfrac{1}{x^2}\log\left(\dfrac{\tan x}{x}\right)$.

$$\therefore\ \lim_{x\to 0}\log y=\lim_{x\to 0}\frac{\log(\tan x/x)}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\dfrac{x}{\tan x}\left(\dfrac{x\sec^2 x-\tan x}{x^2}\right)}{2x}$$

$$=\lim_{x\to 0}\frac{x\sec^2 x-\tan x}{2x^2\tan x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sec^2 x+2x\sec^2 x\tan x-\sec^2 x}{4x\tan x+2x^2\sec^2 x}$$

$$=\lim_{x\to 0}\frac{\sec^2 x\tan x}{2\tan x+x\sec^2 x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sec^2 x+2\sec x(\sec x\tan x)\tan x}{2\sec^2 x+\sec^2 x+2x\sec^2 x\tan x}=\frac{1}{3}.$$

$$\therefore\ \lim_{x\to 0}\log y=\frac{1}{3}\ \Rightarrow\ \lim_{x\to 0}y=e^{1/3}.$$

Hence $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{1/x^2}=e^{1/3}$

11. $\lim\limits_{x\to 0}(1+x)^{1/x}=e$.

Solution: Let $y=(1+x)^{1/x}$, so that $\log y=\dfrac{1}{x}\log(1+x)$.

$$\therefore\ \lim_{x\to 0}\log y=\lim_{x\to 0}\frac{\log(1+x)}{x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{1/(1+x)}{1}=1.$$

$\therefore\ \lim\limits_{x\to 0}\log y=\log\ \lim\limits_{x\to 0}y=1$ Hence, $\lim\limits_{x\to 0}(1+x)^{1/x}=e^1=e$.

12. $\lim\limits_{x\to a}\left(2-\dfrac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}=e^{2/\pi}$, where $a\in\mathbb{R}$.

13. $\lim\limits_{x\to 0}(1+\sin x)^{\cot x}=e$.

Solution: The given limit is 1^∞ form. Let $y=(1+\sin x)^{\cot x}$, so that $\log y=\cot x\log(1+\sin x)$.

$$\therefore\ \lim_{x\to 0}\log y=\lim_{x\to 0}\frac{\log(1+\sin x)}{\tan x}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\cos x}{(1+\sin x)\sec^2 x}=1.$$

Hence $\lim\limits_{x\to 0}y=e^1=e$ or $\lim\limits_{x\to 0}(1+\sin x)^{\cot x}=e$.

7.8. Convex Set

A set S of real numbers is said to be convex if $x, y \in S$, $\lambda \in (0,1)$ implies $\lambda x + (1-\lambda) y \in S$.

7.8.1. Convex Function

Let $S \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$. Then a function $f : S \to \mathbb{R}$ is said to be convex function. If for any $x_1, x_2 \in S$, $f(\lambda x_1 + (1-\lambda) x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$, where $\lambda \in [0,1]$ i.e., Image of convex combination of points of $S \le$ Convex combination of images of corresponding points.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as $f(x) = x^2$. Then f is a convex function.

Reason: As $(x_1 - x_2)^2 \ge 0 \Rightarrow \quad x_1^2 + x_2^2 \ge 2x_1x_2 \Rightarrow 2x_1^2 + 2x_2^2 \ge (x_1 + x_2)^2$

$$\Rightarrow \frac{x_1^2 + x_2^2}{2} \ge \left(\frac{x_1 + x_2}{2}\right)^2 \Rightarrow \frac{1}{2}\left(f(x_1) + f(x_2)\right) \ge f\left(\frac{1}{2}(x_1 + x_2)\right)$$

7.8.2. Concave Function

Let $S \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$. Then a function $f : S \to \mathbb{R}$ is said to be a concave function if for any $x_1, x_2 \in S$, $f(\lambda x_1 + (1-\lambda) x_2) \ge \lambda f(x_1) + (1-\lambda) f(x_2)$, where $\lambda \in [0,1]$ i.e., Image of convex combination of points of $S \ge$ Convex combination of images of corresponding points.

Note:

(a) The condition of convexity is equivalent to the geometric condition that the curve is below (may be identical in special case) the segment joining any two points of the graph of the convex function.

(b) For a concave function the chord should lie below the graph.

(c) Convexity and concavity are geometric characters of a function.

(d) It is not always easy to determine whether a function is convex or concave. But for differentiable function we shall derive a simple criterion to determine this. But clearly if the tangents at every point of a curve lies below curve, the curve is convex and if the tangents lie above, the curve is concave.

(e) A function may be convex on one part of its domain while it is concave or linear on other parts.

(f) If f is a convex function then $-f$ is concave function and conversely.

(g) Constant function is concave as well as convex (infact any straight line is convex as well as concave function)

Result: A differentiable function $f : S \to \mathbb{R}$ is convex function on S $\Leftrightarrow f''(x) \ge 0 \quad \forall\, x \in S$

Result: A differentiable function $f : S \to \mathbb{R}$ is concave function on S $\Leftrightarrow f''(x) \le 0 \quad \forall\, x \in S$

Note: Differentiability is not a necessary condition. *i.e.,* a non differentiable function may be convex

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as $f(x) = |x| \quad \forall\, x \in \mathbb{R}$

Then $f(\lambda x_1 + (1-\lambda) x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$

$(\because |\lambda x_1 + (1-\lambda) x_2| \le \lambda |x_1| + (1-\lambda)|x_2|)$

CHAPTER 8

RIEMANN INTEGRAL

8.1. Definition

Integration is concerned with the problem of finding the area of a region under a cure.

Let us start with a simple problem: Find the area A of the region enclosed by a circle of radius r.

For an arbitrary n, consider the n equal inscribed and super-scribed triangles as shown in Figure 1.

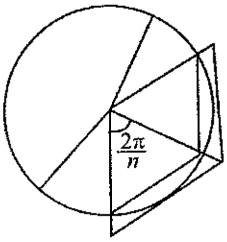


Figure 1

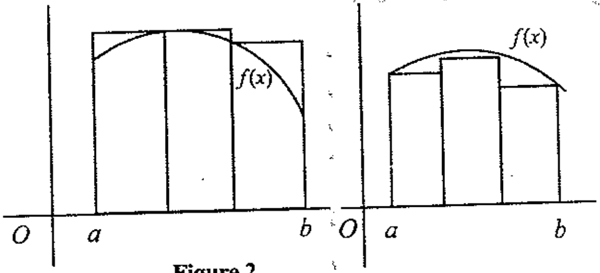


Figure 2

Since A is between the total areas of the inscribed and super-scribed triangles, we have

$$nr^2\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)\le A\le nr^2\tan\left(\frac{\pi}{n}\right)$$

By sandwich theorem, $A=\pi r^2$. We will use this idea to define and evaluate the area of the region under a graph of a function.

Suppose f is a non-negative function defined on the interval $[a,b]$. We first subdivide the interval into a finite number of subintervals. Then we squeeze the area of the region under the graph of f between the areas of the inscribed and super-scribed rectangles constructed over the subintervals as shown in Figure 2. If the total areas of the inscribed and super-scribed rectangles converge to the same limit as we make the partition of $[a,b]$ finer and finer then the area of the region under the graph of f can be defined as this limit and f is said to be integrable.

Let us define whatever has been explained above formally.

Definition: Let $[a,b]$ be a given interval. A partition P of $[a,b]$ is a finite set of points $x_0,x_1,x_2,\dots,x_n$ such that $a=x_0\le x_1\le\dots\le x_{n-1}\le x_n=b$. We write $P=\{x_0,x_1,x_2,\dots,x_n\}$.

If P is a partition of $[a,b]$ we write $\Delta x_i=x_i-x_{i-1}$ for $1\le i\le n$. Let f be a bounded real valued function on $[a,b]$. For a partition P of $[a,b]$, we define

$M_i=\sup\{f(x):x_{i-1}\le x\le x_i\}$ and $m_i=\inf\{f(x):x_{i-1}\le x\le x_i\}$.

$U(P,f)=\sum_1^n M_i\Delta x_i$ and $L(P,f)=\sum_1^n m_i\Delta x_i$.

The numbers $U(P,f)$ & $L(P,f)$ are called upper and lower Riemann sums for the partition P (see Figure 2).

Since f is bounded, there exist real numbers m and M such that $m \le f(x) \le M$, for all $x \in [a,b]$. Thus for every partition P,

$m(b-1) \le L(P,f) \le U(P,f) \le M(b-a)$.

We define

$$\overline{\int_a^b} f(dx) = \inf U(P,f) \qquad \ldots(1)$$

And $$\underline{\int_a^b} f(dx) = \sup L(P,f). \qquad \ldots(2)$$

(1) and (2) are called upper and lower Riemann integrals of f over $[a,b]$ respectively.

If the upper and lower integrals are equal, we say that f is Riemann integrable or integrable.

In this case the common value of (1) and (2) is called the Riemann integral of f and is denoted by $\int_a^b f\,dx$ or $\int_a^b f(x)\,dx$.

***Example* 1:** Consider the function $f:[0,1] \to \mathbb{R}$ defined by

$f\left(\frac{1}{2}\right)=1$ and $f(x)=0$ for all $x \in [0,1] \setminus \left\{\frac{1}{2}\right\}$.

Then f is integrable. We show this using the definition as follows. For any partition P of $[0,1]$, $L(P,f)$ is always 0 and hence the lower integral is 0. Let us evaluate the upper integral. Let $P=\{x_1,x_2,\ldots,x_n\}$ be any partition of $[0,1]$ and $\frac{1}{2} \in [x_i, x_{i+1}]$ for some i. Then $U(P,f) \le 2 \max \Delta x_j$. Since we can always choose a partition P such that $\max \Delta x_j$ is as small as possible, the upper integral, which is the infimum of $U(P,f)'s$, is 0. Hence, f is integrable and $\int_0^1 f(x)\,dx = 0$.

***Example* 2:** Not every bounded function is integrable. For example the function

$f(x)=1$ if x is rational and 0 otherwise

Is not integrable over any interval $[a,b]$ (check this).

In general, determining whether a bounded function on $[a,b]$ is integrable, using the definition, is difficult. For the purpose of checking the integrability, we give a criterion for integrability, called Riemann criterion, which is analogous to the Cauchy criterion for the convergence of a sequence.

Let us define some concepts and results before presenting the criterion. Throughout, we will assume that f is a bounded real function on $[a,b]$.

Definition: A partition P_2 of $[a,b]$ is said to be finer than a partition P_1 if $P_2 \supset P_1$. In this case we say that P_2 is a refinement of P_1. Given two partition P_1 and P_2, the partition $P_1 \cup P_2 = P$ is called their common refinement.

The following theorem illustrates that refining partition increases lower terms and decreases upper terms.

***Theorem* 1:** Let P_2 be a refinement of P_1 then $L(P_1,f) \le L(P_2,f)$ and $U(P_2,f) \le U(P_1,f)$.

Proof: First we assume that P_2 contains just one more point than P_1. Let this extra point be x^*. Suppose $x_{i-1} < x^* < x_i$, where $x_i - 1$ and x_i are consecutive points of P_1. Let

$w_1 = \inf\{f(x) : x_{i-1} \le x \le x^*\}$ and

$w_2 = \inf\{f(x) : x^* \le x \le x_i.\}$

Then $w_1 \ge m_i$ and $w_2 \ge m_i$ where $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$. Then

$L(P_2,f) - L(P_1,f) = w_1(x^* - x_{i-1}) + w_2(x_i - x^*) - m_i(x_i - x_{i-1}) \ge 0.$

If P_2 contains k more points then we repeat this process k-times. The other inequality is analogously proved. (Prove it).

The geometric interpretation suggests that the lower integral is less than or equal to the upper integral. So the next result is also anticipated.

Corollary 2: $\overline{\int_a^b} f\,dx \ge \underline{\int_a^b} f\,dx.$

Proof: Let P_1, P_2 be two partitions and let P be their common refinement. Then

$L(P_1,f) \le L(P,f) \le U(P,f) \le (P_2,f).$

Thus for any two partitions P_1 and P_2, we have $L(P_1,f) \le U(P_2,f)$.

Fix P_2 and take sup over all P_1. Then $\underline{\int_a^b} f\,dx \le U(P_2,f)$. Now take inf over all P_2 to get the desired result.

In the following result we present the Riemann criterion (a necessary and sufficient condition for the existence of the integral of a bounded function).

***Theorem* 3:** (Riemann's criterion for integrability): f is integrable on $[a,b] \Leftrightarrow$ for every $\varepsilon > 0$ there exists a partition P such that

$$U(P,f) - L(P,f) < \varepsilon. \qquad \ldots(3)$$

Proof: For any P, we have

$$L(P,f) \le \underline{\int_a^b} f\,dx \le \overline{\int_a^b} f\,dx \le U(P,f).$$

Therefore (3) implies that

$$\overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx < \varepsilon, \quad \forall \varepsilon > 0.$$

Hence $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$ i.e. f is integrable. Conversely, suppose f is integrable and $\varepsilon > 0$. Then there exist partitions P_1 and P_2. Then $U(P,f) - L(P,f) < \varepsilon$.

The proof of the following corollary is immediate from the previous theorem.

Corollary 3: Let $f:[a,b] \to \mathbb{R}$ be a bounded function. Suppose (P_n) is a sequence of partitions of $[a,b]$ such that the sequence $\{\|P_n\|\}$ converges to 0, and $U(P_n,f) - L(P_n,f) \to 0$, then f is integrable.

Problem: Let $f:[0,1] \to \mathbb{R}$ such that $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$ Show that f is integrable and find $\int_0^1 f(x)\,dx$.

Solution: We will use the Riemann criterion to show that f is integrable on $[0,1]$. Let $\varepsilon > 0$ be given. We will choose a partition P such that $U(P,f) - L(P,f) < \varepsilon$. Since $\frac{1}{n} \to 0$, there exists N such that $1/n \in [0,\varepsilon]$ for all $n > N$. So only finite number of $\frac{1'}{n}s$ lies in the interval $[\varepsilon,1]$. Cover these finite number of $\frac{1'}{n}s$ by the intervals $[x_1,x_2],[x_3,x_4],\ldots,[x_{m-1},x_m]$ such that $x_i \in [\varepsilon,1]$ for all $i = 1,2,\ldots,m$ and the sum of the length of these m intervals is less than ε. Consider the partition $P = \{x_1,x_2,\ldots,x_m\}$. It is clear that $U(P,f) - L(P,f) < 2\varepsilon$. Hence by the Reimann criterion the function is integrable. Since the lower integral is 0 and the function is integrable, $\int_0^1 f(x)\,dx = 0$.

We will apply the Riemann criterion for integrability to prove the following two existence theorems.

***Theorem* 4:** If f is continuous on $[a,b]$ then f is integrable.

Proof: Let $\varepsilon > 0$. Since f is uniformly continuous, choose $\delta > 0$ such that $|s-t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon$ for $s,t \in [a,b]$.

Let P be a partition of $[a,b]$ such that $\Delta x_i < \delta \ \forall\, i = 1,2,\ldots,n$. Then

$M_i - m_i \le \varepsilon \ \forall i = 1,2,\ldots,n$.

Hence

$$U(P,f) - L(P,f) = \sum_1^n (M_i - m_i)\,\Delta x_i \le \varepsilon(b-a).$$

This implies that f is integrable.

***Theorem* 5:** If f is a monotone function on $[a,b]$ then f integrable.

Proof: Suppose f is monotonically increasing (the proof is similar in the other case)

Choose a partition P such that $\Delta x_i = \frac{b-a}{n}$. Then $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Therefore

$$U(P,f)-L(P,f) = \frac{b-a}{n}\sum_1^n \left[f(x_i)-f(x_{i-1})\right]$$

$$= \frac{b-a}{n}\left[f(b)-f(a)\right]$$

$< \varepsilon$ for large n.

Hence f is integrable.

In the following problem we will see that limit and integral cannot be interchanged.

Problem: Let $g_n(y) = \begin{cases} \frac{ny^{n-1}}{1+y} & \text{if } 0 \le y < 1 \\ 0 & \text{if } y = 1 \end{cases}$ Then prove that $\lim_{n\to\infty}\int_0^1 g_n(y)\,dy = \frac{1}{2}$ whereas $\int_0^1 \lim_{n\to\infty} g_n(y)\,dy = 0$.

Solution: From the ratio test for sequences we can show that $\lim_{n\to\infty}\frac{ny^{n-1}}{1+y} = 0$, for each $0 < y < 1$. Therefore $\int_0^1 \lim_{n\to\infty} g_n(y)\,dy = 0$.

For the other part, use integration by parts to see that $\int_0^1 \frac{ny^{n-1}}{1+y}dy = \frac{1}{2}+\int_0^1 \frac{y^n}{(1+y)^2}dy$. Note that $\int_0^1 \frac{y^n}{(1+y)^2}dy \le \int_0^1 y^n = \frac{1}{n+1} \to 0$ as $n \to \infty$. Therefore, $\lim_{n\to\infty}\int_0^1 g_n(y)\,dy = \frac{1}{2}$.

8.2. Quick Review for Riemann Integrability

1. If a bounded function f is Riemann integrable on $[a,b]$ then $\int_a^b f\,dx = \int_a^{\overline{b}} f\,dx = \int_{\underline{a}}^b f\,dx$
2. Every bounded function need not be Riemann integrable.
3. If M,m are bounds of a bounded function f over $[a,b]$, then $m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$.
4. If $f(x) = \begin{cases} 0, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$, f is not integrable on any bounded interval but $f(x)$ bounded.
5. Every constant function is integrable.

6. Let f be a bounded function on $[a,b]$ and P a partition of $[a,b]$. If P^* is any refinement of P, then

   (i) $U(P^*,f) \le U(P,f)$

   (ii) $L(P^*,f) \ge L(P,f)$

7. Let P^* be a refinement of P containing r points more than P, where P is a partition of $[a,b]$ with norm $\mu(P) < \delta$ and $|f(x)| < k \,\forall\, x \in [a,b]$. Then

   (i) $U(P^*,f) \ge U(P,f) - 2rk\delta$

   (ii) $L(P^*,f) \le L(P,f) + 2rk\delta$

8. If P_1 and P_2 are any two partitions of $[a,b]$, then $L(P_1,f) \le U(P_2,f)$ i.e. an upper sum can never be less than any lower sum.

9. If M, m are the bounds of an integrable function f over $[a,b]$, then $m(b-a) < \int_a^b f\,dx \le M(b-a)$.

10. If f a bounded function on $[a,b]$ then for each $\varepsilon > 0$ there exists $\delta \ge 0$ such that

    (a) $U(P,f) < \int_a^{\bar{b}} f\,dx + \varepsilon$ (b) $U(P,f) > \int_a^{\bar{b}} f\,dx - \varepsilon$

    For all partitions P of $[a,b]$ with norm $\mu(P) < \delta$

11. If f is continuous and integrable on $[a,b]$, then there exists a number c lying between a and b such that $\int_a^b f\,dx = (b-a)f(c)$.

12. If f is bounded and integrable on $[a,b]$ such that $|f(x)| < k \,\forall\, x \in [a,b]$, then $\left|\int_a^b f\,dx\right| < k|b-a| \;\; k > 0$.

13. Necessary and sufficient condition for the Integrability of a bounded function f is that to every $\varepsilon > 0$, there corresponds $\delta > 0$ such that for every partition P of $[a,b]$ with norm $\mu(P) < \delta$, $U(P,f) - L(P,f) < \varepsilon$.

14. A necessary and sufficient condition for a bounded function f to be integrable over $[a,b]$ is that $\lim_{\mu(P)\to 0} w(P,f) = 0$, where $w(p,f)$ is oscillatory sum of $[a,\ b]$.

15. Every continuous function on $[a,b]$ is integrable on $[a,b]$.

16. An integrable function on $[a,b]$ need not be continuous function.

17. Every monotonic function f on $[a,b]$ is integrable on $[a,b]$.

18. A bounded function f having only finite number of points of discontinuity in $[a,b]$ is integrable on $[a,b]$

19. If a function f is bounded in $[a,b]$ and the set of its points of discontinuity has a finite number of limit points, then f is integrable in $[a,b]$.

20. If a bounded function f is integrable at $[a,b]$ implies f is also integrable on $[a,c]$ and $[c,b]$, $a<c<b$ and $\int_a^b f\,dx=\int_a^c f\,dx+\int_c^b f\,dx$.

21. If f is bounded and integrable on $[a,c],[c,b]$ then f is integrable over $[a,b]$ and $\int_c^b f\,dx=\int_a^c f\,dx+\int_c^b f\,dx$.

22. If f_1 and f_2 are two bounded and integrable functions on $[a,b]$, then

(i) f_1+f_2 is bounded and integrable on $[a,b]$ and $\int_a^b (f_1+f_2)\,dx=\int_a^b f_1\,dx+\int_a^b f_2\,dx$.

(ii) f_1-f_2 is bounded and integrable on $[a,b]$ and $\int_a^b (f_1-f_2)\,dx=\int_a^b f_1\,dx-\int_a^b f_2\,dx$.

23. If f is integrable on $[a,b]$, then f^2 is also integrable on $[a,b]$.

24. If f_1 and f_2 are both integrable on $[a,b]$, then $f_1 f_2$ is also integrable on $[a,\ b]$.

25. If f_1 and f_2 are two bounded and integrable functions on $[a,b]$ and let $\lambda>0$ such that $|f_2(x)|>\lambda$ for all x in $[a,b]$, then $\frac{f_1}{f_2}$ is bounded and integrable on $[a,b]$.

26. If f is bounded and integrable on $[a,b]$, then $|f|$ is also bounded and integrable on $[a,b]$ and $\left|\int_a^b f\,dx\right|\le\int_a^b |f|\,dx$.

27. If $|f|$ is integrable, then f need not be integrable.

28. A function f is bounded and integrable on $[a,b]$ and there exists a function F such that $F'=f$ on $[a,b]$, and $\int_a^b f\,dx=F(b)-F(a)$.

29. If f and g are integrable on $[a,b]$ and g keeps the same sign over $[a,b]$ then there exists a number μ lying between the bounds of f such that $\int_a^b f\,g\,dx=\mu\int_a^b g\,dx$.

30. If $\int_a^b \phi\,dx$ and $\int_a^b g\,dx$ both exists and ϕ is positive and monotonically decreasing on $[a,b]$, then there exists a point $\xi\in[a,b]$ such that $\int_a^b \phi\,dx=\phi(a)\int_a^{\xi} g\,dx$.

31. If $\int_a^b f\,dx$ and $\int_a^b g\,dx$ both exist and f is monotonic on $[a,b]$, then there exists some $\xi\in[a,b]$ such that $\int_a^b fg\,dx=f(a)\int_a^{\xi} g\,dx+f(b)\int_{\xi}^b g\,dx$.

Put Your Own Notes

32. If f is a non-negative continuous function on $[a,b]$ such that $\int_a^b f dx=0$, then $f(x)=0, \forall x\in[a,b]$.

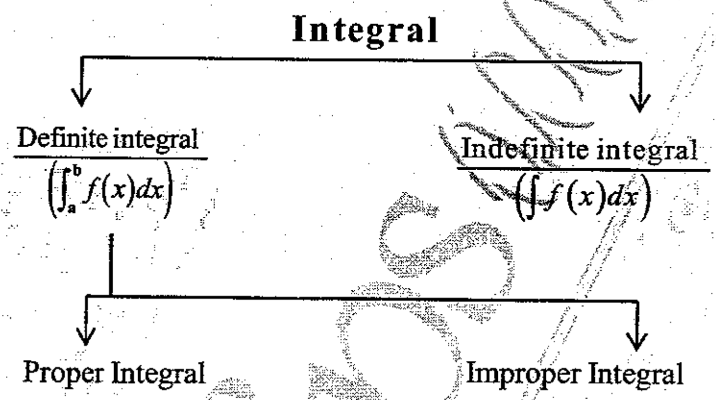
33. Let f be bounded integrable on $[a,b]$ then $\int_a^b [f(x)]^2 dx=0 \Leftrightarrow f(c)=0$ at every point c of continuity of f.

34. If the function f and ϕ are bounded and integrable in $[a,b]$ then $\left[\int_a^b f(x)\phi(x)dx\right]^2 \le \int_a^b [f(x)]^2 dx . \int_a^b [\phi(x)]^2 dx$ and equality hold only if f & g are constant.

35. Let f be integrable on $[a,b]$ and for each $x\in[a,b]$, let $F(x)=\int_a^x f(t)dt$ then F is uniformly continuous on $[a,b]$.

36. If f is bounded and integrable on $[a, b]$, then $F(x)=\int_a^x f(t)dt$, $(a\le x\le b)$ is continuous on $[a,b]$ and If f is continuous on $[a, b]$ then $F'=f$.

8.3. Improper Integral



$\int_a^b f(x)dx$ is said to be proper integrals if $[a,b]$ If either of a or b is ∞ or $-\infty$ but $f(x)$ is bounded

Finite & $f(x)$ is bounded on $[a,b]$ such type of integral or a & b are finite but $f(x)$ is unbounded is always convergent.

Introduction

In the chapter of Riemann Integral the theory of integration was developed under two assumptions ._

(i) The intervall of integration was required to be a closed & bounded interval.

(ii) The integrand was required to be bounded on the interval.

- The scope of the theory of integration may be widened by relaxing these restrictions. If these restrictions are relaxed we have the following two type of integrals, called improper integrals or infinite integrals.

(a.) Improper integrals on a finite interval where the integrand is unbounded.

(b.) Improper integrals on an unbounded interval.

Now we define convergence of improper integrals & discuss the properties of each type separately_

A. Improper integrals on a closed & bounded interval, the integrand having infinite discontinuities:

Point of infinite discontinuity

A point $x=a$ is called point of infinite discontinuity (P.I.D.) if $\lim_{x\to a} f(x)=+\infty$ or $-\infty$.

1. Convergence at the left end point

(i) Let the left end point 'a' of the closed & bounded interval $[a,b]$ be the only point of infinite discontinuity of a function $f(x)$ which is bounded and integrable on $[a+\in,b]$ for every $\in$ satisfying $0<\in<b-a$.

The we say that improper integral $\int_a^b f(x)dx$ exists, or converges at a; if $\lim_{\in\to 0^+}\int_{a+\in}^b f(x)dx$ exists.

Thus, $\int_a^b f(x)dx=\lim_{\in\to 0^+}\int_{a+\in}^b f(x)dx$; provided that limit on R.H.S. exists.

If the limit does not exist, we say that improper integral $\int_a^b f(x)dx$ does not exists or is divergent.

Example (i) : $\int_0^1 f(x)dx=\int_0^1 \frac{1}{x}dx$; here '0' is the only point of infinite discontinuity of the integrand.

The intergrand is bounded & integrable on $[0+\in,1]$: for all $\in$ satisfying $0<\in<1$.

$\lim_{\in\to 0^+}\int_{0+\in}^1 \frac{1}{x}dx=\lim_{\in\to 0^+}[-\log\in]=\infty$

Therefore, the improper integral $\int_0^1 \frac{1}{x}dx$ is divergent.

(ii) $\int_0^1 \frac{1}{\sqrt{x}}dx$; 0 is a point of infinite discontinuity of the integrand.

The integrand is bounded and integrable on $[0+\in,1]$; for all $\in$ satisfying $0<\in<1$.

$\lim_{\in\to 0^+}\int_{0+\in}^1 \frac{1}{\sqrt{x}}dx=\lim_{\in\to 0^+}\left[2-2\sqrt{\in}\right]=2$

Therefore, the improper integral $\int_{0+\in}^1 \frac{1}{\sqrt{x}}dx$ is convergent & $\int_0^1 \frac{1}{\sqrt{x}}dx=2$.

2. Convergence at the right end point

Let the right end point 'b' of the closed & bounded interval $[a,b]$ be the only point of infinite discontinuity of a function $f(x)$ which is bounded & integrable on $[a,b-\in]$ for every $\in$ satisfying $0<\in<b-a$.

Then we say that the improper integral $\int_a^b f(x)dx$ exists, or converges at b , if $\lim_{\in\to0^-}\int_a^{b-\in} f(x)dx$ exists.

Thus , $\int_a^b f(x)dx=\lim_{\in\to0^+}\int_a^{b-\in} f(x)dx$; provided that the limit on R.H.S. exists.

If the limit does not exists, we say the improper integral $\int_a^b f(x)dx$ does not exists or is divergent.

Example:

(i) The integral $\int_0^1 \frac{1}{1-x}dx$ is improper , since 1 is a point of infinite discontinuity of the integrand.

Clearly, the integrand is bounded and integrable on $[0,1-\in]$ for all $\in$ satisfying $0<\in<1$.

$$\lim_{\in\to0^+}\int_0^{1-\in}\frac{1}{1-x}dx=\lim_{\in\to0^+}[-\log\in]=\infty.$$

Therefore, the improper integral $\int_0^1\frac{1}{1-x}dx$ is divergent.

(ii) $\int_0^1\frac{1}{\sqrt{1-x^2}}dx$; 1 is a point of infinite discontinuity of the integrand.

Clealy, the integrand is bounded & integrable on $[0,1-\in]$ for all $\in$ satisfying $0<\in<1$.

$$\lim_{\in\to0^+}\int_0^{1-\in}\frac{1}{\sqrt{1-x^2}}dx=\lim_{\in\to0^+}\left[\sin^{-1}(1-\in)\right]=\frac{\pi}{2}.$$

Therefore, the improper integral $\int_0^1\frac{1}{\sqrt{1-x^2}}dx$ is convergent & $\int_0^1\frac{1}{\sqrt{1-x^2}}dx=\frac{\pi}{2}$.

3. Convergence at the both end point

Let the end points $a\&b$ be the only points of infinite discontinuity of a function $f(x)$ which is bounded & integrable on $[a+\in,b-\in']$ for every $\in,\in'$ satisfying $0<\in,\in'<b-a$.

Let c be any point such that $a<c<b$. If the improper integrals and $\int_c^b f(x)dx$ both converge at 'a' & 'b' respectively. Then we say the improper integral $\int_a^b f(x)dx$ converges &write $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$.

Example: $\int_0^2 \frac{1}{\sqrt{x(2-x)}}$ is improper integral , since 0&2 are points of infinite discontinuity of the integrand.

Clearly, the integrand is bounded & integrable on $[0+\in, 2-\in']$ for all $\in,\in'$ satisfying $0<\in 2, 0<\in'<2$.

$$\lim_{\in\to 0^+}\int_{0+\in}^{1}\frac{1}{\sqrt{x(2-x)}}dx=\lim_{\in\to 0^+}\left[\sin^{-1}(x-1)\right]_{0+\in}^{1}=\frac{\pi}{2} \qquad \&$$

$$\lim_{\in'\to 0^+}\int_{1}^{2-\in'}\frac{1}{\sqrt{x(2-x)}}dx=\lim_{\in'\to 0^-}\left[\sin^{-1}(x-1)\right]_{1}^{2-\in'}=\frac{\pi}{2}$$

Therefore, $\int_0^2 \frac{1}{\sqrt{x(2-x)}}dx$ is convergent & $\int_0^2 \frac{1}{\sqrt{x(2-x)}}dx=\pi$.

4. **Convergence at Any of the mid point:**

Let a point $c(a\le c\le b)$ of the closed & bonded interval $[a,b]$ be the only point of infinite discontinuity of a function $f(x)$.

If the improper integrals $\int_a^c f(x)dx \& \int_c^b f(x)dx$ be both convergent according to the definitions given above, then the improper integral $\int_a^b f(x)dx$ is said to be convergent & we write

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$$

Therefore, if both the limits $\lim_{\in\to 0^+}\int_a^{c-\in} f(x)dx \& \lim_{\in\to 0^+}\int_{c-\in}^{a} f(x)dx$ exists then

$$\int_a^b f(x)dx=\lim_{\in\to 0^+}\int_a^{c-\in} f(x)dx+\lim_{\in\to 0^+}\int_{c-\in}^{b} f(x)dx.$$

If the improper integral $\int_a^b f(x)dx$ is convergent, its value is also equal to the symmetric limit $\left[\lim_{\in\to 0^+}\int_a^{c-\in} f(x)dx+\lim_{\in\to 0^+}\int_{c-\in}^{b} f(x)dx\right]$.

Note: It may happen that the improper integral $\int_a^b f(x)dx$ is divergent but the limit $\lim_{\in\to 0^+}\left[\int_a^{c-\in} f(x)dx+\int_{c+\in}^{b} f(x)dx\right]$ exists, then this symmetric limit is called the Cauchy principal value of the improper integral $\int_a^b f(x)dx$ and it is denoted by $P\int_a^b f(x)dx$.

Example: $\int_{-1}^{1} f(x)dx$; where $f(x)=\begin{cases}\frac{1}{x}; & x\ne 0\\ 0; & x=0\end{cases}$.

Here, $\lim_{\in\to 0^+}\int_{-1}^{0-\in} f(x)dx+\lim_{\in\to 0^+}\int_{0+\in}^{1} f(x)dx$

$=\lim_{\in\to 0^+}\log\in+\lim_{\in'\to 0^+}(-\log\in')$ & this limit does not exist if $\in\to 0^+, \in'\to 0^+$ independently.

But $\lim_{\in\to 0^+}\left[\int_{-1}^{0-\in} f(x)dx+\int_{0+\in}^{1} f(x)dx\right]=\lim_{\in\to 0^+}(\log\in-\log\in)=0$

Therefore, the integral $\int_{-1}^{1} f(x)dx$ is divergent but $p\int_{-1}^{1} f(x)dx=0$.

B. Improper Integrals On An Unbounded Interval:

1. The improper integral $\int_a^\infty f(x)dx$; where f is integrable on $[a,t];t>a$ and , $\int_a^\infty f(x)dx=\lim_{t\to\infty}\int_a^t f(x)dx$; provided that limit exists .

   If limit exist finitely then the improper integral $\int_a^\infty f(x)dx$ is said to be convergent. If limit does not exist finitely then the improper integral $\int_a^\infty f(x)dx$ is said to be divergent.

   Example: $\int_a^\infty f(x)dx=\lim_{t\to\infty}\int_0^t e^{-x}dx=\lim_{t\to\infty}\int_0^t\left(1-e^{-t}\right)=1$

   Therefore , $\int_0^\infty e^{-x}dx$ is convergent & $\int_0^\infty e^{-x}dx=1$.

   Similarly , $\int_0^\infty \frac{1}{1+x}dx$ is divergent.

2. The improper integral $\int_{-\infty}^b f(x)dx$; where f is integrable on $[t,b]$; $t<b$ & $\int_{-\infty}^b f(x)dx=\lim_{t\to-\infty}\int_{-\infty}^b f(x)dx$; provided that limit exists. If limit exist finitely then $\int_{-\infty}^b f(x)dx$ is convergent otherwise $\int_{-\infty}^a f(x)$ is said to be divergent .

3. Convergence of the improper integral $\int_{-\infty}^\infty f(x)dx$; where f is integrable on $[t_1,t_2];t_1,t_2\in\mathbb{R}$ such that $t_1<t_2$.

   let $c\in\mathbb{R}$. If both the integrals $\int_{-\infty}^c f(x)dx\&\int_c^\infty f(x)dx$ be convergent according to the definition I & II above, then the improper integral $\int_{-\infty}^\infty f(x)dx$ is said to be convergent & we write $\int_{-\infty}^\infty f(x)dx=\int_{-\infty}^c f(x)dx+\int_c^\infty f(x)dx$.

   Example: $\int_{-\infty}^\infty \frac{1}{1+x^2}dx$; the integrand is integrable on any closed interval $[t_1,t_2];t_1<t_2$.

   $$\int_{-\infty}^a \frac{1}{1+x^2}dx+\int_a^\infty \frac{1}{1+x^2}dx;a\in\mathbb{R}.$$

   $$\int_t^a \frac{dx}{1+x^2}=\tan^{-1}a-\tan^{-1}t\Rightarrow\lim_{t\to-\infty}\left(\tan^{-1}a-\tan^{-1}t\right)=\tan^{-1}a+\frac{\pi}{2}.$$

$$\& \int_a^t \frac{dx}{1+x^2} = \tan^{-1} t - \tan^{-1} a \Rightarrow \lim_{t\to\infty}\left(\tan^{-1} a - \tan^{-1} t\right) = \frac{\pi}{2} - \tan^{-1} a$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \left(\tan^{-1} a + \frac{\pi}{2}\right) + \left(\frac{\pi}{2} - \tan^{-1} a\right) = \pi$$

Thus, $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is convergent.

Comparison Test

1. Let $I_1 = \int_a^b f(x)dx; I_2 = \int_a^b g(x)dx$ be two improper integrals where.

   (i) f and g have only P.I.D. at $x = a$

   (ii) f and g are non $-ve$

   (iii) $f(x) \le g(x); \forall x \in [a,b]$

   Then

   (a) If $\int_a^b g(x)dx$ is convergent then $\int_a^b f(x)dx$ is convergent.

   (b) If $\int_a^b f(x)dx$ is divergent then $\int_a^b g(x)dx$ is divergent.

2. Now if $b \to \infty$ then all the result is same for $\int_a^{\infty} f(x)dx$.

Limit Form Test of Comparison Test:

1. If $I_1 = \int_a^b f(x)dx; I_2 = \int_a^b g(x)dx$ be two improper integrals . Where f, g are define on $[a,b] \& x = a$ is only point of infinite discontinuity such that__

   (i) $f \& g$ are non $-ve$

   (ii) $\lim_{x\to a} \frac{f(x)}{g(x)} = \ell$ (finite & non- zero)

   Then both integrals converge or diverge together.

2. Now, if $b \to \infty$ then all the result is same for $\int_a^{\infty} f(x)dx$ using $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \ell$; in $[a,t]$.

   Note: If $\ell = 0$ & $\int_a^b g(x)dx$ convergent then $\int_a^b f(x)dx$ convergent & if $\ell = \infty$ & $\int_a^b g(x)dx$ divergent then $\int_a^b f(x)dx$ divergent.

Results:

(i) $\int_0^1 \frac{1}{x^P}dx$ is convergent if $P<1$ and is divergent if $P \geq 1$

(ii) $\underset{(a>0)}{\int_a^\infty} \frac{1}{x^P}dx$ is convergent if $P>1$ and is divergent if $P \leq 1$

(iii) $\int_a^b \frac{1}{(x-a)P}dx$ is convergent if $P<1$ and is divergent if $P \geq 1$

(iv) $\int_a^b \frac{1}{(b-x)^P}dx$ is convergent if $P<1$ and is divergent if $P \geq 1$

Example: $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x^P}dx$

$$\because \frac{\sin x}{x^P} = \frac{\sin x}{x} \cdot \frac{1}{x^{P-1}} \leq \frac{1}{x^{P-1}}$$

$$\Rightarrow \underset{(f)}{\frac{\sin x}{x^P}} \leq \underset{(g)}{\frac{1}{x^{P-1}}} \quad \text{and 0 is only P.I.D.}$$

$\int_0^{\frac{\pi}{2}} \frac{1}{x^{P-1}}dx$ is convergent if $P-1<1 \Rightarrow P<2$

$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{x^P}$ convergent if $P<2$.

By limit form test: $\lim\limits_{x\to 0} \frac{\frac{\sin x}{x^P}}{\frac{1}{x^{P-1}}} = 1$ (finite and non-zero)

$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{1}{x^{P-1}}dx \;\&\; \int_0^{\frac{\pi}{2}} \frac{\sin x}{x^P}dx$ behaves a like

$\Rightarrow \frac{1}{x^{p-1}}$ is convergent for $p-1<1 \Rightarrow p<2 \Rightarrow \int_0^{\pi/2} \frac{\sin x}{x^p}dx$ Convergent if $p<2$.

$\Rightarrow \frac{1}{x^{p-1}}$ is divergent for $p-1 \geq 1 \Rightarrow p \geq 2 \Rightarrow \int_0^{\pi/2} \frac{\sin x}{x^p}dx$ Divergent if $p \geq 2$

Absolutely convergent: The improper integral $\int_a^\infty f(x)dx$ is said to be absolutely convergent if the integral $\int_a^\infty |f(x)|dx$ is convergent.

Definition is same if $a \to -\infty$

- The improper integral which is convergent but not absolutely convergent in called conditionally convergent.

Example: $(x)=\begin{cases}\frac{1}{x^2};x\in Q\\ \frac{-1}{x^2};x\in Q^c\end{cases}$ & $f(x)$ is defined on $[1,\infty]$

$\Rightarrow|f(x)|=\frac{1}{x^2};x\in[1,\infty]\Rightarrow\int_1^\infty|f(x)|dx$ convergent

$\Rightarrow\int_1^\infty f(x)dx$ is A.C. $\Rightarrow\int_1^\infty f(x)dx$ convergent

Test for Convergence of the Integrals of a Product:

1. **Let a function** f be integrable on $[a,X];\forall\times>a$ & the integral $\int_a^\infty f(x)dx$ is absolutely convergent & a function ϕ be bounded on $[a,\infty]$ & integrable on $[a,X];\forall\times>a$.

   Then the integral $\int_a^\infty f(x).\phi(x)dx$ is absolutely convergent.

2. **Abel's Test:**

   Let

   (i) a function ϕ be monotonic & bd on $[a,\infty]$ &

   (ii) The integral $\int_a^\infty f(x)dx$ be convergent

   Then the integral $\int_a^\infty f(x).\phi(x)dx$ is convergent.

3. **Dirichlet's test:**

   Let

   (i) a function ϕ be monotonic & bd on $[a,\infty]$ & $\lim_{x\to\infty}\phi(x)dx=0$

   (ii) The integral $\int_a^X f(x)dx$ be bd on $[a,X];\forall\times>a$.

   Then the integral $\int_a^\infty f(x)\phi(x)dx$ in convergent.

Example: $\int_0^\infty\frac{\sin^3x}{x^5}dx=\int_a^\infty\sin^3x.\frac{1}{x^5}dx$

Let $f=\frac{1}{x^5};g=\sin^3x$

Here, $\left|\int_0^t\sin^3xdx\right|\le m;\forall t\in[a,\infty]$

$\lim_{x\to}\frac{1}{x^5}=0$; then $\int_0^\infty\frac{\sin^3x}{x^5}$ is convergent

Cauchy – Maclaurin Integral Test: If f in a monotonic diseasing function on $[1,\infty]$ & $f(x)>0 \ \forall x\in[1,\infty]$, then the improper integral $\int_1^\infty f(x)dx$ & the infinite series $\sum_1^\infty f(x)$ converge on diverge together.

Beta Function: The integral $\int_0^1 x^{m-1}(x-x)^{n-1}dx; m>0, n>0$ is called the Beta function & it is denoted by $B(m,n)$.

Thus, $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx; m>0; n>0$

The improper integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent if $m>0, n>0$.

Gamma Function: The integral $\int_0^\infty e^{-x}x^{n-1}dx; n>0$ is called the gamma function & it is denoted by $\overline{|(x)}$.

Thus, $\overline{|(x)}=\int_0^\infty e^{-x}x^{n-1}dx; n>0$

The improper integral $\int_0^\infty e^{-x}x^{n-1}dx$ is convergent if $n>0$.

Result: $\int_0^\infty \frac{\sin mx}{x^n}dx(m>0,n>0)$ is convergent if $0<n<2$ & absolutely convergent if $1<n<2$.

- The improper integral $\int_1^\infty \sin x^p dx$ is convergent if $p>1$
- The improper integral $\int_1^\infty \frac{\sin x}{x^p}dx$ is convergent absolutely if $p>1$
  - The improper integral $\int_0^\infty \frac{\sin x^m}{x^n}dx$ convergent if $n+m>1$

μ-Test: Let a be the only point of infinite discontinuity of a function f which is integrable on $[a+\in,b]$ for $0<\in<b-a$ & $f(x)>0$; $\forall x\in(a,b]$ if $\lim_{x\to a} f(x)(x-a)^\mu=\ell;(\ell\neq0)$ then $\int_a^b f(x)dx$ in convergent iff $\mu<1$.

Note: If $\lim_{x\to b^-} f(x)(b-x)^\mu=\ell$; where ℓ is a finite non –zero no. then $\int_a^b f(x)dx$ is convergent iff $\mu<1$.

CHAPTER 9

UNIFORM CONVERGENCE

9.1. Definition

Previously, we have studied sequences of real numbers. Now we discuss the topic of sequence of real valued functions. A sequence of functions $\{f_n\}$ is a list of functions $(f_1, f_2, \ldots)$ such that each f_n maps a given subset D of $\mathbb{R}$ into $\mathbb{R}$.

1. Pointwise convergence

Definition. Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of functions defined on D. We say that $\{f_n\}$ converges pointwise on D if $\lim\limits_{n\to\infty} f_n(x)$ exists for each point x in D.

If $\{f_n\}$ is pointwise convergent then the function defined by $f(x) = \lim\limits_{n\to\infty} f_n(x)$, for every x in D, is called the pointwise limit of the sequence $\{f_n\}$.

***Example* 1:** Let $\{f_n\}$ be the sequence of functions on $\mathbb{R}$ defined by $f_n(x) = nx$. This sequence does not converge pointwise on $\mathbb{R}$ because $\lim\limits_{n\to\infty} f_n(x) = \infty$ for any $x > 0$.

***Example* 2:** Let $\{f_n\}$ be the sequence of functions on $\mathbb{R}$ defined by $f_n(x) = x/n$. This sequence converges pointwise to the zero function on $\mathbb{R}$.

***Example* 3:** Consider the sequence $\{f_n\}$ of functions defined by

$$f_n(x) = \frac{nx + x^2}{n^2} \text{ for all } x \text{ in } \mathbb{R}.$$

Show that $\{f_n\}$ converges pointwise.

Solution: For every real number x, we have :

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} + \frac{x^2}{n^2} = x\left(\lim_{n\to\infty} \frac{1}{n}\right) + x^2\left(\lim_{n\to\infty} \frac{1}{n^2}\right) = 0 + 0 = 0$$

Thus, $\{f_n\}$ converges pointwise to the zero function on $\mathbb{R}$.

***Example* 4:** Consider the sequence $\{f_n\}$ of functions defined by

$$f_n(x) = \frac{\sin(nx+3)}{\sqrt{n+1}} \text{ for all } x \text{ in } \mathbb{R}.$$

Show that $\{f_n\}$ converges pointwise.

Solution: For every x in $\mathbb{R}$, we have

$$\frac{-1}{\sqrt{n+1}} \le \frac{\sin(nx+3)}{\sqrt{n+1}} \le \frac{1}{\sqrt{n+1}}$$

Moreover,

$$\lim_{n\to\infty}\frac{1}{\sqrt{n+1}}=0.$$

Applying the squeeze theorem for sequences, we obtain that

$\lim\limits_{n\to\infty} f_n(x)=0$ for all x in $\mathbb{R}$.

Therefore, $\{f_n\}$ converges pointwise to the function $f\equiv 0$ on $\mathbb{R}$.

***Example* 5:** Consider the sequence $\{f_n\}$ of functions defined by $f_n(x)=n^2x^n$ for $0\le x\le 1$. Determine whether $\{f_n\}$ is pointwise convergent.

Solution: First of all, observe that $f_n(0)=0$ for every n in $\mathbb{N}$. So the sequence $\{f_n(0)\}$ is constant and converges to zero. Now suppose $0<x<1$ then $n^2x^n=n^2e^{n\ln(x)}$. But $\ln(x)<0$ when $0<x<1$, it follows that

$\lim\limits_{n\to\infty} f_n(x)=0$ for $0<x<1$

Finally, $f_n(1)=n^2$ for all n. So, $\lim\limits_{n\to\infty} f_n(1)=\infty$. Therefore, $\{f_n\}$ is not pointwise convergent on $[0,1]$.

***Example* 6:** Let $\{f_n\}$ be the sequence of functions defined by $f_n(x)=\cos^n(x)$ for $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$. Discuss the pointwise convergence of the sequence.

Solution: For, $-\frac{\pi}{2}\le x<0$ and for $0<x\le\frac{\pi}{2}$, we have

$0<\cos(x)<1$.

It follows that

$\lim\limits_{n\to\infty}(\cos(x))^n=0$ for $x\ne 0$.

Moreover, since $f_n(0)=1$ for all n in $\mathbb{N}$, one gets $\lim\limits_{n\to\infty} f_n(0)=1$.

Therefore, $\{f_n\}$ converges pointwise to the function f defined by

$$f(x)=\begin{cases}0 & \text{if } -\frac{\pi}{2}\le x<0 \text{ or } 0<x\le\frac{\pi}{2}\\ 1 & \text{if } x=0\end{cases}$$

***Example* 7:** Consider the sequence of function defined by

$f_n(x)=nx(1-x)^n$ on $[0,1]$.

Show that $\{f_n\}$ converges pointwise to the zero function.

Solution: Note that $f_n(0)=f_n(1)=0$, for all $n\in\mathbb{N}$. Now suppose $0<x<1$, then

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty} nxe^{n\ln(1-x)}=x\lim_{n\to\infty} ne^{n\ln(1-x)}=0$$

Because $\ln(1-x)<0$ when $0<x<1$. Therefore, the given sequence converges pointwise to zero.

***Example* 8:** Let $\{f_n\}$ be the sequence of functions on $\mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n^3 & \text{if } \ 0 < x \leq \dfrac{1}{n} \\ 1 & \text{otherwise} \end{cases}$$

Show that $\{f_n\}$ converges pointwise to the constant function $f = 1$ on $\mathbb{R}$.

Solution: For any x in $\mathbb{R}$ there is a natural number N such that x does not belong to the interval $(0, 1/N)$. The intervals $(0, 1/n)$ get smaller as $n \to \infty$. Therefore, $f_n(x) = 1$ for all $n > N$. Hence,

$$\lim_{n\to\infty} f_n(x) = 1 \quad \text{for all } x.$$

For Formal Definition of Pointwise Convergence

Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ converges ponitwise to f if given any x in D and given any $\varepsilon > 0$, there exists a natural number $N = N(x, \varepsilon)$ such that

$$| f_n(x) - f(x) | < \varepsilon \ \text{ for every } n > N.$$

Note: The notation $N = N(x, \varepsilon)$ means that the natural number N depends on the choice of x and ε.

2. **Uniform convergence**

Definition: Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of real valued functions defined on D. Then $\{f_n\}$ converges uniformly to f if given any $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that

$$| f_n(x) - f(x) | < \varepsilon \ \text{ for every } n > N \text{ and for every } x \text{ in } D.$$

Note: In the above definition the natural number N depends only on ε. Therefore, uniform convergence implies pointwise convergence. But the converse is false as we can see from the following counter-example.

***Example* 9:** Let $\{f_n\}$ be the sequence of functions on $(0, \infty)$ defined by

$$f_n(x) = \frac{nx}{1 + n^2x^2}.$$

This function converges pointwise to zero. Indeed, $(1 + n^2x^2) \sim n^2x^2$ as n gets larger and larger. So,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{n^2x^2} = \frac{1}{x} \lim_{n\to\infty} \frac{1}{n} = 0.$$

But for any $\varepsilon < \frac{1}{2}$, we have

$$\left| f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right) \right| = \frac{1}{2} - 0 > \varepsilon.$$

Hence $\{f_n\}$ is not uniformly convergent.

Theorem: Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of continuous function on D which converges uniformly to f on D. Then f is continuous on D

Problem 1: Let $\{f_n\}$ be the sequence of functions on $[0,1]$ defined by $f_n(x)=nx(1-x^4)^n$. Show that $\{f_n\}$ converges pointwise. Find its pointwise limit.

Problem 2: Is the sequence of functions on $(0,1]$ defined by $f_n(x)=(1-x)^{\frac{1}{n}}$ pointwise convergent? Justify your answer.

Problem 3: Consider the sequence $\{f_n\}$ of functions defined by

$$f_n(x)=\frac{n+\cos(nx)}{2n+1} \text{ for all } x \text{ in } \mathbb{R}.$$

Show that $\{f_n\}$ is pointwise convergent. Find its pointwise limit.

Problem 4: Consider the sequence $\{f_n\}$ of functions defined on $[0,\pi]$ by $f_n(x)=\sin^n(x)$. Show that $\{f_n\}$ converges pointwise. Find its pointwise limit. Using the above theorem, show that $\{f_n\}$ is not uniformly convergent.

9.2. Quick Review for Uniform Convergence

1. The sequence of function x^n converges uniformly to 0 on $[0,a]$ where $0<a<1$, but not on $[0,1)$.
2. If $f_n(x)$ converges point-wise to $f(x)$ on S and $M_n=\sup\limits_{x\in S}|f_n(x)-f(x)|$, then $f_n(x)$ converges uniformly to $f(x)$ on S iff $M_n\to 0$, as $n\to\infty$.
3. A sequence of functions $f_n(x)$ defined on a set S converges uniformly on S iff for each $\varepsilon>0$ $\exists\, m\in\mathbb{N}$ such that $|f_{n+p}(x)-f_n(x)|<\varepsilon$ $\forall\, n\ge m$, $p\ge 0$ and $x\in S$.
4. The sequence x^n converges on $[0,1]$ to 0 if $0<x<1$ and to 1 iff $x=1$. However, the convergence is not uniform on $[0,1]$.
5. A sequence $f_n(x)$ defined on a set S is said to be **uniformly bounded** on S if there exists $k>0$ such that $|f_n(x)|<k$ $\forall\, x\in S$ and $n\in\mathbb{N}$.
6. If the sequence of continuous functions $f_n(x)$ converges to a continuous function $f(x)$ on a compact set E and $f_n(x)\ge f_{n+1}(x)$ $\forall\, n\in\mathbb{N}$ and $x\in E$, then $f_n(x)\to f(x)$ uniformly on E.
7. If $f_n(x)$ converges uniformly to a function $f(x)$ on S, and x be a limit point of S, and $\lim\limits_{t\to x} f_n(t)=a_n$, then a_n converges and $\lim\limits_{t\to x} f(t)=\lim\limits_{n\to\infty} a_n$.

8. If $f_n(x)$ converges uniformly to a function $f(x)$ on a set S and if each $f_n(x)$ is continuous at $c \in S$ then $f(x)$ is continuous at c on S.

9. $\langle \tan^{-1} nx \rangle$ is uniformly convergent on $[a,b], a>0$, but it is only point-wise convergent on $(0,b]$.

10. If $f_n(x)$ is monotonically non-decreasing sequence of continuous functions on $[a,b]$ and point-wise converges, then the convergence is uniform on $[a,b]$.

11. If $f_n(x)$ is monotonically non-decreasing sequence of continuous functions on $[a,b]$ and the sum function $\sum f_n(x)$ is continuous on $[a,b]$, then $\sum f_n(x)$ is uniformly convergent on $[a,b]$.

12. If $f(x)$ is continuous on $[a,b]$, then there exists a sequence of polynomials $P_n(x)$ such that $P_n(x)$ converges uniformly to $f(x)$ on $[a,b]$.

13. If $f_n(x)$ is a sequence of integrable functions on $[a,b]$ such that it converges uniformly to a function $f(x)$ on $[a,b]$, then $f(x)$ is also integrable on $[a,b]$ and $\lim\limits_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$. Moreover, $\int_a^x f_n(t)\,dt$ converges to $\int_a^x f(t)\,dt$ uniformly on $[a,b]$.

14. If $f_n(x)$ converges for a point in $[a,b]$ and $f_n'(x)$ converges uniformly on $[a,b]$, then $f_n(x)$ converges uniformly to a function $f(x)$ on $[a,b]$, and $f'(x) = \lim\limits_{n\to\infty} f_n'(x) \;\; \forall\, x \in [a,b]$.

15. If $f_n(x), g_n(x)$ converge uniformly to $f(x), g(x)$ respectively on a set S, then $f_n(x) \pm g_n(x)$ converge uniformly to $f(x) \pm g(x)$ respectively on S. Also $k f_n(x)$ converges uniformly to $k f(x)$ on S, where $k \in \mathbb{R}$.

16. If $f_n(x), g_n(x)$ converges uniformly on a set S then the product sequence $f_n(x) g_n(x)$ may, or may not converge uniformly on S.

17. If the sequence of monotonically non-decreasing continuous functions $f(x)$ converges point-wise to a continuous function $f(x)$ on $[a,b]$, then the convergence is uniform on $[a,b]$.

18. If $f_n(x) \to f(x)$ uniformly on $[a,b]$, where each $f_n(x)$ and $g(x)$ are continuous on $[a,b]$, then

$$\lim_{n\to\infty}\int_a^b f_n(x)\,g(x)\,dx = \int_a^b f(x)\,g(x)\,dx.$$

19. If each $f_n(x)$ is integrable on $[a,b]$ and $f_n(x) \to f(x)$ uniformly on $[a,b]$ thenv $f(x)$ is integrable on $[a,b]$ and $\lim\limits_{n\to\infty}\int_a^b f_n(x)\,dx$ exists, and is equal to $\int_a^b f(x)\,dx$.

20. If $f_n \to f$ uniformly on a set S and $x \in S'$ so as each $\lim f_n$ as $t \to x$ exists on S, then $\lim f$ as $t \to x$ exists on S and $\lim\limits_{t\to x} f(t) = \lim\limits_{n\to\infty} \lim\limits_{t\to x} f_n(t)$ on S.

21. If $f_n(x)$ converges for a point in $[a,b]$ and $f_n'(x)$ converges uniformly on $[a,b]$, then $f_n(x)$ converges uniformly to a function $f(x)$ on $[a,b]$ and $f'(x) = \lim\limits_{n\to\infty} f_n'(x)\ \forall\, x \in [a,b]$.

22. $\sum\limits_{n=1}^{\infty} f_n(x)$ converges uniformly on S iff for every $\varepsilon > 0\ \exists\, m \in \mathbb{N}$ such that $\left|f_{n+1}(x) + f_{n+2}(x) + \ldots + f_{n+p}(x)\right| < \varepsilon\ \ \forall\ n \ge m,\ p \ge 0$ and $x \in S$.

23. If $\sum f_n(x) \to f(x)$ uniformly on a set S and x be a limit point of S and $\lim\limits_{t\to x} f_n(t) = a_n$, then $\sum a_n$ converges and $\lim\limits_{t\to x} f(t) = \sum\limits_{n=1}^{\infty} a_n$.

24. If $f_n(x)$ are continuous at $c \in S$ and $\sum f_n(x) \to f(x)$ uniformly on S, then $f(x)$ is continuous at $c \in S$.

25. If each $f_n(x) \in R[a,b]$ and $\sum f_n(x)$ converges uniformly on $[a,b]$, then $\int_a^b \sum\limits_{n=1}^{\infty} f_n(x)\,dx = \sum\limits_{n=1}^{\infty} \int_a^b f_n(x)\,dx$.

26. If $f_n(x) \in R[a,b]\ \forall\, n$ and $\sum f_n(x)$ converges uniformly to $f(x)$ on $[a,b]$, then $\sum\limits_{n=1}^{\infty} \int_a^x f_n(x)\,dx$ converges uniformly to $\int_a^x f(x)\,dx$ on $[a,b]$.

27. If $f_n(x) \in R[a,b]\ \forall\, n$ and $\sum f_n(x)$ converges uniformly to $f(x)$ on $[a,b]$ and $g(x) \in R[a,b]$, then $\sum\limits_{n=1}^{\infty} \int_a^x g(x) f_n(x)\,dx$ converges uniformly to $\int_a^x g(x)\, f(x)\,dx$ on $[a,b]$

28. If each $f_n(x)$ is differentiable on $[a,b]$ and $\sum f_n'(x)$ converges uniformly on $[a,b]$ and $\sum f_n(x)$ converges for some point $c \in [a,b]$, then $\sum f_n(x)$ converges uniformly on $[a,b]$ and $s'(x) = \sum f_n'(x)$ on $[a,b]$, where $s(x) = \sum f_n(x)$.

29. If $f_n(x)$ has continuous derivatives on $[a,b]$ and $\sum f_n(x), \sum f_n'(x)$ converge uniformly to $f(x)$ and $g(x)$ respectively on $[a,b]$, then $f(x)$ is differentiable and $f'(x) = g(x)$ on $[a,b]$.

30. If $|f_n(x)| \le M_n\ \ \forall\, x \in S$ and $n \in \mathbb{N}$, then $\sum f_n(x)$ converges uniformly (and absolutely) on S if $\sum M_n$ converges.

31. The series $\sum 3^n \sin \dfrac{1}{4^n} x$ converges absolutely and uniformly on (a, ∞), where $a > 0$.

Put Your Own Notes

32. The function $f(x)=\sum_{n=1}^{\infty} b^n \cos\left(a^n \pi x\right)$ where $0<b<1$ and a is an odd positive integer such that $ab>1+\frac{3}{2}\pi$, is everywhere continuous but it has no derivative for any $x \in \mathbb{R}$.

33. If $\sum f_n(x)$ converges uniformly on a set S, then $f_n(x)$ converge uniformly to 0 on S.

34. If $\sum f_n(x)$, $\sum g_n(x)$ converge uniformly to $f(x), g(x)$ respectively on a set S, then $\sum\left(f_n(x) \pm g_n(x)\right)$ converge uniformly to $f(x) \pm g(x)$ respectively on S also $\sum k\, f_n(x)$ will converge uniformly to $k\, f(x)$ on S, where k is any constant real number.

35. $\sum x^n$ converges uniformly on $(-\delta,\delta)$, where $1>\delta>0$, but not on $(-1,1)$.

36. If $\sum f_n(x)$ converges uniformly to $f(x)$ on $[a,b]$ and each $f_n(x) \in R[a,b]$, then $f(x) \in R[a,b]$ and $\int_a^b f(x)\,dx = \sum \int_a^b f_n(x)\,dx$.

37. If each $f_n'(x)$ is continuous on $[a,b]$ and $\sum f_n(x), \sum f_n'(x)$ converge uniformly to $s(x)$ and $g(x)$ respectively on $[a,b]$ then, $s'(x)$ exists and $s'(x)=g(x)$ on $[a,b]$.

38. If $\sum f_n(x)$ converges uniformly on a set S and $g_n(x)$ is monotonic and uniformly bounded on S, then the series $\sum f_n(x) g_n(x)$ converges uniformly on S.

39. If $\sum f_n(x)$ is uniformly bounded on S and $g_n(x)$ is monotonic and converges uniformly to zero on S, then $\sum f_n(x)\, g_n(x)$ converges uniformly on S.

40. If $\sum f_n(x)$ converges uniformly to $f(x)$ on a set S and $\sum a_n$ converges to a, then $\sum\left(f_n(x)+a_n\right)$ converges uniformly to $f(x)+a$ on S.

CHAPTER 10

FUNCTION OF SEVERAL VARIABLES

10.1. Definition of n-tuples

The Euclidean n-space $\mathbb{R}^n$ is the set of all n-tuples $(x_1,...,x_n)$ of real numbers $x_i, i=1,..,n$ (called vectors or points) on which the two operations of vector addition and scaler multiplication are defined as follows:

(i) For any two points $(x_1,...,x_n)$ and $(y_1,...,y_n)$ in $\mathbb{R}^n$

$$(x_1,...,x_n)+(y_1,...,y_n)=(x_1+y_1,...,x_n+y_n)$$

(ii) For any $a \in \mathbb{R}$ and $(x_1,...,x_n) \in \mathbb{R}^n$

$$a\cdot(x_1,...,x_n)=(ax_1,...,ax_n)$$

we can write $a\cdot(x_1,...,x_n)=a(x_1,...,x_n)$. If $x \in \mathbb{R}^n$ then $x=(x_1,...,x_n)$ for some real numbers $x_1,...,x_n$, and x_i is called the i^{th} component of x.

Neighbourhood: Let $S \subseteq \mathbb{R}^n$ and $\alpha=(a_1,a_2,...,a_n) \in \mathbb{R}^n$. Then we say S is neighbourhood of α, if $\exists \varepsilon > 0$ such that $B(\alpha,\varepsilon)=\left\{(x_1,x_2,...,x_n)\in\mathbb{R}^n : \sqrt{(x_1-a_1)^2+(x_2-a_2)^2+...+(x_n-a_n)^2}<\varepsilon\right\}\subseteq S$

Open Set: Let $S \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. Then S is said to be open if it is neighborhood of each of its point.

Closed Set: Let $S \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. Then S is said to be closed if complement of S in $\mathbb{R}^n$ is open set i.e., $\mathbb{R}^n - S$ is an open set.

Remarks:

(a) An open Ball in $\mathbb{R}^n$ is an open set.

(b) A closed ball in $\mathbb{R}^n$ is closed set.

(c) The two sets $\mathbb{R}^n$ and ϕ are both open and closed sets in $\mathbb{R}^n$ and moreover there is no other subsets of $\mathbb{R}^n$ which is both open and closed.

Important Results:

1. The union of any collection of open sets in $\mathbb{R}^n$ is open.
2. The intersection of a finite collection of open sets in $\mathbb{R}^n$ is open.
3. The intersection of any collection of closed sets in $\mathbb{R}^n$ is closed.
4. The union of finite collection of closed sets is closed.
5. A subset A of $\mathbb{R}^n$ is open iff A is the union of a countable collection of open rectangles (balls).

6. A subset A of $\mathbb{R}^n$ is closed iff A is the intersection of a countable collection of closed sets.

7. Let A be a subset of $\mathbb{R}^n$ and w be a point in $\mathbb{R}^n$. The translation of A by w is denoted by $w+A$ and is defined by $w+A \equiv \{w+v : v \in A\}$, Then
   (i) A is open if and only if $w+A$ is open.
   (ii) A is closed if and only if $w+A$ is closed.

   Cluster Point: A point $x \in \mathbb{R}^n$ is called a cluster point of $A \subset \mathbb{R}^n$ if every open ball containing x has a nonempty intersection with $A-\{x\}$.

   If $x \in A$ but it is not a cluster point of A then x is called an isolated point of A i.e. $x \in w \cap A - \{x\} \neq \phi,\ x \in A,\ w \cap A = \{x\}$.

8. A subset of $\mathbb{R}^n$ is closed iff the subset contains all its cluster points.

   Product of Sets: Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$. The product of $A \& B$ denoted by $A \times B$ is defined as $A \times B = \{(x_1,\ldots,x_n,y_1,\ldots,y_m) : (x_1,\ldots,x_n) \in A \,\&\, (y_1,\ldots,y_m) \in B\}$. If $x \in A \,\&\, y \in B$ then the point $(x_1,\ldots,x_n,y_1,\ldots,y_m)$ in $A \times B$ is denoted by (x,y).

   From definition it is evident that $A \times B \subset \mathbb{R}^{n+m}$, and in particular $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$.

9. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$. Then
   (i) $A \& B$ are open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively iff $A \times B$ is open in $\mathbb{R}^{n+m}$.
   (ii) A and B are closed sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively iff $A \times B$ is closed in $\mathbb{R}^{n+m}$.

10. Let $A, B \subseteq \mathbb{R}^n$. Then
   (i) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
   (ii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

11. If $A \subset \mathbb{R}^n$, and $B \subset \mathbb{R}^n$, and x is a cluster point of $A \cap B$ then x is a cluster point of both $A \& B$.

12. If $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^n$, and x is a cluster point of $A \cup B$ then x is a cluster point of A or of B.

10.2. Real Valued Function on n Variables

Let $S \subseteq \mathbb{R}^n$. Then a map $f : S \to \mathbb{R}$ is called real valued function on n variables $x_1, x_2, \ldots x_n$. Provided no two different images are mapped under same point in S i.e. for $x, y \in S, f(x) \neq f(y) \Rightarrow x \neq y$.

Limit of a real valued function on n variables:

Let $S \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}$ be a function. Let $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$ be limit point of S and $l \in \mathbb{R}$. Then we say limit of f as $x = (x_1, x_2, \ldots x_n) \to (\alpha_1, \alpha_2, \ldots \alpha_n)$ is l and we write $\lim\limits_{x \to \alpha} f(x_1, x_2, \ldots x_n) = l$, if for each $\varepsilon > 0, \exists\ \delta > 0$ such that $|f(x_1, x_2, \ldots x_n) - l| < \varepsilon$, whenever $0 < \sqrt{(x_1-\alpha_1)^2 + \ldots + (x_n-\alpha_n)^2} < \delta$.

13. Let $f \& g$ be two real valued function with common domain $A \subset \mathbb{R}^n$, and c a cluster point of A. If $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exists, then

(i) $\lim\limits_{x\to c}(f+g)(x) = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x)$

(ii) $\lim\limits_{x\to c}(f\cdot g)(x) = \lim\limits_{x\to c} f(x) \cdot \lim\limits_{x\to c} g(x)$

(iii) $\lim\limits_{x\to c}\left(\dfrac{f}{g}\right)(x) = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$ provided $g \&$ its limit at c not vanish.

Continuity of a Function at a Point:

Let $f: D \to \mathbb{R}$, where D is an open subset of $\mathbb{R}^n$. Then f is said to be continuous at a point $a = (a_1, a_2, ..., a_n) \in D$ if $\lim\limits_{x\to a} f(x) = f(a)$, where $x = (x_1, x_2, ...x_n)$.

In other words, f is continuous at a point $(a_1, a_2, ..., a_n) \in D$ if given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(x_1, x_2, ...x_n) - f(a_1, a_2, ..., a_n)| < \varepsilon$, when $\sqrt{\left[(x_1 - a_1)^2 + ... + (x_n - a_n)^2\right]} < \delta$.

Partial Derivatives: If f is a function of n variables, then its partial derivatives are the functions f_{x_i}'s defined by $f_{x_i}(x_1, x_2, ...x_n) = \lim\limits_{h\to 0} \dfrac{f(x_1, ..., x_i + h, ...x_n) - f(x_1, x_2, ...x_n)}{h}$ for all $1 \le i \le n$.

14. Let f and g are functions of n variables. Also if $f_{x_i}, f_{x_j}, g_{x_i}, g_{x_j}$ exist. Then

(i) $(f \pm g)_{x_i} = f_{x_i} \pm g_{x_i}$

(ii) $(f.g)_{x_i} = f_{x_i} g + g_{x_i} f$

(iii) $\left(\dfrac{f}{g}\right)_{x_i} = \dfrac{g \cdot f_{x_i} - f \cdot g_{x_i}}{g^2}$

Differentiability at a Point: Let $f: D \to \mathbb{R}$ be a real-valued function defined on an open set $D \subset \mathbb{R}^2$. Then f is said to be differentiable at a point $(a,b) \in D$, if $f(a+h, b+k) - f(a,b) = Ah + Bk + \sqrt{h^2 + k^2}\, g(h,k)$,Where A and B are real numbers independent of h and k, and g is a real-valued function such that $\lim\limits_{(h,k)\to(0,0)} g(h,k) = 0$.

Equivalently, f is differentiable at (a,b), if $f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi_1(h,k) + k\phi_2(h,k)$ where A and B are real numbers independent of h and k and $\phi(h,k)$, $\phi_1(h,k)$ and $\phi_2(h,k)$ converges to 0 as $(h,k) \to (0,0)$.

Remark:

(i) It may be noted that the constants A, B appearing in the definition of differentiability of f are given by : $A = f_x(a,b)$, $B = f_y(a,b)$

(ii) If either of f_x, f_y does not exist at (a,b), then f is not differentiable at (a,b).

15. If a function $f : D \to \mathbb{R}$ is differentiable at $(a,b) \in D$, then f has partial derivatives f_x and f_y at (a,b).

16. If a function $f : D \to \mathbb{R}$ is differentiable at a point $(a,b) \in D$, then f is continuous at (a,b).

Remark: The converse of the above theorem may not be true. A function may be continuous at a point without being differentiable at that point.

17. If (a,b) be a point of the domain of definition of a function f such that

(i) f_x is continuous at (a,b)

(ii) f_y exists at (a,b),

Then f is differentiable at (a,b).

Note: In a similar way it can be shown that f is differentiable at (a,b), if f_x exists and f_y is continuous at (a,b). In fact, one of the partial derivatives is to be continuous and the other merely to exist at the point.

Remark:

(i) If a function f is not differentiable at (a,b), then f_x and f_y cannot be continuous at (a,b).

(ii) The conditions of the above theorem are sufficient, but are not necessary. If a function f is differentiable at a point (a,b), then f_x, f_y may not be continuous at (a,b).

Chain Rule 1: Suppose that $z = f(x,y)$ is a differentiable function of $x \& y$, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

Chain Rule 2: Suppose that $z = f(x,y)$ is a differentiable function of x and y, and where $x = g(s,t)$, $y = h(s,t)$ and the partial derivatives g_x, g_t, h_s and h_t exist. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Directional Derivatives: Let f be a function of n variables defined on a disk D centered at $(x_1, x_2, ... x_n)$ and let $u = (a_1, a_2, ..., a_n)$ be a unit vector. Then the directional derivative of f at $(x_1, x_2, ... x_n)$ in the direction of u, denoted $D_u f(x_1, x_2, ... x_n)$ is defined by $\lim\limits_{h \to 0} \dfrac{f(x_1 + ha_1, x_2 + ha_2, ..., x_n + ha_n) - f(x_1, x_2, ... x_n)}{h}$ If limit exists.

18. A Sufficient Condition for Continuity:

A sufficient condition for a function f to be continuous at (a,b) is that one of the partial derivatives exists and is bounded in a neighbourhood of (a,b) and that the other exists at (a,b).

19. If f is a differentiable function of x and y then f has a directional derivative in the direction of any unit vector $u = (a,b)$ and $D_u f(x,y) = f_x(x,y)a + f_y(x,y)b = \nabla f(x,y) \cdot u$.

Remark: If the unit vector u makes an angle θ with the positive x-axis then we can write $u = (\cos\theta, \sin\theta)$ and $D_u f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$.

Function from $\mathbb{R}^n$ to $\mathbb{R}^m$: A map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined as $f(x_1, x_2, ..., x_n) = (\phi_1, \phi_2, ..., \phi_m)$ is called function in n-variables to m-variables, where for each $i = 1, 2, ..., m$, $\phi_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a real valued function in n-variables.

10.3. Limit and Continuity of Vectors and Real Valued Functions

(i) **Definition (Limit):** Let $f : A \to \mathbb{R}^m, A \subset \mathbb{R}^n$ and c is a cluster point of A. Then f is said to have a limit l in $\mathbb{R}^m$ at the point c in $\mathbb{R}^n$ if for every $\varepsilon > 0, \exists \delta > 0$ such that $\forall x \in A$ $0 < |x - c| < \delta \Rightarrow |f(x) - l| < \varepsilon$. $\lim\limits_{x \to c} f(x) = l$

(ii) **Definition (Continuity):** A function $f : A \to \mathbb{R}^m, A \subset \mathbb{R}^n$ is said to be continuous at a point $a \in A$ if a is not a cluster point of A or $\lim\limits_{x \to c} f(x) = f(a)$.

(iii) **Definition (Uniform continuity):** Given a function $f : A \to \mathbb{R}^m, f$ is said to be uniformly continuous on A iff $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x, y \in A : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

(iv) **Definition (Diffrentiability):** Let A be an open set in $\mathbb{R}^n$ and $f : A \to \mathbb{R}^m$, then f is said to be differentiable at $a \in A$ if and only if there is a linear transformation $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that $\lim\limits_{h \to 0} \dfrac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0$ If f is differentiable at every point of A, then f is said to be a differentiable function.

20. Let $f:A\to\mathbb{R}^m$, where $A\subset\mathbb{R}^n$ and c is cluster point of A then $\lim\limits_{x\to c}f(x)=l$ iff $\lim\limits_{x\to c}\phi_i(x)=l_i$ $i=1,...,m$.

21. Suppose $f:A\to\mathbb{R}, A\subset\mathbb{R}^n$ and c is a cluster point of A. If $\lim\limits_{x\to c}f(x)$ exists, then $\exists M>0,\delta>0$ such that $\forall x\in A$ $0<|x-c|<\delta\Rightarrow|f(x)|<M$.

22. If $f:A\to\mathbb{R}^m, A\subset\mathbb{R}^n$ and c is a cluster point of A such that $\lim\limits_{x\to c}f(x)$ exists then $\lim\limits_{x\to c}|f(x)|$ exists and further more $\lim\limits_{x\to c}|f(x)|=\left|\lim\limits_{x\to c}f(x)\right|$.

23. Let $f:A\to\mathbb{R}^m, A\subset\mathbb{R}^n$ and $a\in A$, then f is continuous at a iff $\phi_i, i=1,...,m$ is continuous at a .

24. Let $f,g:A\to\mathbb{R}^m, A\subset\mathbb{R}^n$ and $\varphi:A\to\mathbb{R}$. If f,g and φ are continuous at $a\in A$, then $f+g, f\cdot g$ and $\varphi\cdot f$ are all continuous at $a\in A$.

25. If $T:\mathbb{R}^n\to\mathbb{R}^m$ is a linear transformation, then T is continuous function.

26. Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a linear transformation, then f is differentiable.

27. Let $f:A\to\mathbb{R}^m$ and $g:B\to\mathbb{R}^k, A\subset\mathbb{R}^n$ and $B\subset\mathbb{R}^m$ such that $B\supseteq$ range f . If $\lim\limits_{x\to a}f(x)=b\in B$ and g is continuous at b then $\lim\limits_{x\to a}(gof)(x)=g(b)$.

28. Let $f:A\to\mathbb{R}^m$, $g:B\to\mathbb{R}^k, A\subset\mathbb{R}^m$ and such that range $f\subset B$. Suppose f is continuous at $a\in A$, and g is continuous at $f(a)\in B$. Then gof is continuous at a .

29. Let $f:A\to\mathbb{R}^m, A\subset\mathbb{R}^n$, then f is continuous on A iff for every open set $U\subset\mathbb{R}^m$ there is an open set $V\subset\mathbb{R}^n$ such that $f^{-1}(U)=V\cap A$.

30. Let $f:A\to\mathbb{R}^m, A\subset\mathbb{R}^n$, then f is continuous on A iff for every closed set $F\subset\mathbb{R}^m$ there is a closed set $H\subset\mathbb{R}^n$ such that $f^{-1}(F)=H\cap A$.

31. Let m and n be positive integers then the limit $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^ny^m}{x^2+y^2}$ Exists if and only if $m+n>2$.

32. Let $A\subseteq\mathbb{R}^n$ be a subset and $f:A\to\mathbb{R}^m$ be a function differentiable at $a\in A$, then f is continuous at a.

33. Let $A\subseteq\mathbb{R}^n$ be open set and $f:A\to\mathbb{R}^m$ be a function defined as $f(x_1,x_2,...,x_n)=(\phi_1,\phi_2,...,\phi_m)$, where $\phi_i:\mathbb{R}^n\to\mathbb{R}$ be real valued function $\forall i=1,2,...,m$.Then f is differentiable at $a\in A\Leftrightarrow$ each ϕ_i is differentiable at a and its derivative is given by $f'(a)=(\phi_1'(a),\phi_2'(a),...,\phi_m'(a))$.

Jacobian Matrix: Let $f: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ defined by $f(x_1, x_2, \ldots, x_n) = (\phi_1, \phi_2, \ldots, \phi_m)$ Where $\phi_i : \mathbb{R}^n \to \mathbb{R}$ is a real valued function, where $1 \le i \le m$. Then matrix of derivative at $x = c$ is denoted by $D(f(c))$ and it is defined as

$$D(f(c)) = \begin{vmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial\phi_m}{\partial x_1} & \frac{\partial\phi_m}{\partial x_2} & \cdots & \frac{\partial\phi_m}{\partial x_n} \end{vmatrix}_{m\times n} = (a_{ij}) \qquad a_{ij} = \frac{\partial\phi_i}{\partial x_j} \begin{pmatrix} 1 \le i \le m \\ 1 \le j \le n \end{pmatrix}$$

10.4. Maxima and Minima of Functions of Two Variables

Definition:

(i) A function $f(x,y)$ is said to have a maximum at a point (a,b) if there exists aneighbourhood N of (a,b) such that $f(a+h,b+k) < f(a,b) \forall (a+h,b+k) \in N - \{(a,b)\}$

The set $N - \{(a,b)\}$ is called a deleted neighbourhood of (a,b).

(ii) A function $f(x,y)$ is said to have a minimum at a point (a,b) if there exists a neighbourhood N of (a,b) such that $f(a+h,b+k) > f(a,b) \forall (a+h,b+k) \in N - \{(a,b)\}$

(iii) A function $f(x,y)$ is said to have an extremum at a point (a,b) if it has either a maximum or minimum at (a,b).

(iv) A point (a,b) is said to be a critical point for a function $f(x,y)$ if f_x and f_y both exist at (a,b) and $f_x(a,b) = 0$ and $f_y = (a,b) = 0$

(a) If a function $f(x,y)$ has an extremum at any point (a,b) of the domain of f if f admits partial derivatives f_x and f_y at (a,b), then $f_x(a,b) = 0$ and $f_y = (a,b) = 0$

Remark: The conditions of the above theorem are only necessary but are not sufficient.

(b) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a real valued function in two variables which posses continuous second order partial derivatives and $f_{xx}(a,b) = A, f_{xy}(a,b) = B, f_{yy}(a,b) = C$ then

(i) f has maximum at (a,b) if $AC - B^2 > 0, A > 0$.

(ii) f has maximum at (a,b) if $AC - B^2 > 0, A < 0$.

(iii) f has neither maximum value nor minimum value at (a,b) if $AC - B^2 < 0$.

(iv) f may or may not have an extremum at (a,b) if $AC - B^2 = 0$

(This case is often called the doubtful case, which needs further investigation)

10.5. Compactness of Connectedness

1. A collection O of open sets is an open cover of A if every point x in A is in some open set in the collection O.

2. A set $A \subset \mathbb{R}^n$ is called compact if every open cover O of A has a finite sub cover of A.

3. **(Heine-Borel Theorem):** $A \subset \mathbb{R}^n$ is compact iff A is closed and bounded.

4. If $B \subset \mathbb{R}^m$ is compact and $x \in \mathbb{R}^n$ then $\{x\} \times B$ is compact in $\mathbb{R}^{n+m}$.

5. If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are compact then $A \times B \subset \mathbb{R}^{n+m}$ is compact.

6. If $A_i \subset \mathbb{R}^{n_i}, i = 1, ..., k$ are compact then $A_1 \times A_2 \times ... \times A_k \subset \mathbb{R}^{n_1+\cdots+n_k}$ is compact.

   In particular any closed rectangle $W \subset \mathbb{R}^k, k \in \mathbb{N}$ is compact.

7. **(Bolzano-Weierstrass Theorem):** Every bounded infinite subset of $\mathbb{R}^n$ has a cluster point.

8. **(Cantor Intersection Theorem):** Suppose F_1 is a nonempty, closed and bounded subset of $\mathbb{R}^n$, and $\{F_i\}_{i \in \mathbb{N}}$ is a sequence of nonempty closed sets in $\mathbb{R}^n$ such that $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$, Then $\bigcap_{i \in \mathbb{N}} F_i \neq \phi$.

9. A set $A \subset B \subset \mathbb{R}^n$ is said to be dense in $B \subset \mathbb{R}^n$ if $\overline{A} \supset B$. If A is dense in $\mathbb{R}^n$ it is simply called a dense set.

10. If $\{E_n : n \in \mathbb{N}\}$ is a collection of dense open set, then $\bigcap_{n \in \mathbb{N}} E_n$ is nonempty.

11. The set $\mathbb{Q}$ of rational numbers is not the intersection of a countable collection of open sets.

12. The set of all irrational numbers $\mathbb{Q}^c$ is not the union of a countable collection of closed sets in $\mathbb{R}$.

13. If $f : A \to \mathbb{R}^m, A \subset \mathbb{R}^n$ is continuous and A is compact then $f(A)$ is compact.

14. **Extremum Value Theorem:** Let $A \subset \mathbb{R}^n$ be compact. If $F : A \to \mathbb{R}$ is continuous then f takes on a maximum and minimum value on A.

15. Let $A \subset \mathbb{R}^n$ be compact and if $f : A \to \mathbb{R}^m$ is continuous and one to one then the inverse function is continuous.

16. Let $A \subset \mathbb{R}^n$ be compact and if $f : A \to \mathbb{R}^m$ is continuous then f is uniformly continuous.

17. Let $A \subseteq \mathbb{R}^n$ be compact and B be closed subset of A then B be closed and compact subset of A.

18. Let C be closed and K be compact in $\mathbb{R}^n$ then $K + C$ is closed.

Put Your Own Notes

19. Let A, B be compact subsets of $\mathbb{R}^n$ then their sum $A+B$ is compact.

20. Let $A \subseteq \mathbb{R}^n$ and U, V open subsets of $\mathbb{R}^n$ then $U \& V$ are said to be a separation of A if

(i) $U \cap A \neq \phi$

(ii) $V \cap A \neq \phi$

(iii) $(U \cap A) \cap (V \cap A) = \phi$

(iv) $(U \cap A) \cup (V \cap A) = A$

21. A subset A of $\mathbb{R}^n$ is said to be a disconnected set if it has a separation U and V. A is called connected if it is not disconnected.

22. A set $C \subseteq \mathbb{R}$ is connected iff it is an interval.

23. Let $A \subseteq \mathbb{R}^n$ be a connected set and if $f : A \to \mathbb{R}^m$ is continuous then $f(A)$ is connected.

24. In $\mathbb{R}^n$, the only subsets that are open and closed are ϕ and $\mathbb{R}^n$.

25. Let $\{A_i : i \in I\}$, where I is index set, be a collection of connected subsets of a (metric) space X with the property that for all $i, j \in I$ we have $A_i \cap A_j \neq \phi$, then $A = \bigcup_{i \in I} A_i$ is connected.

26. Let A be a countable subset of $\mathbb{R}^n$, $n > 1$, then $\mathbb{R}^n - A$ is connected.

27. Let X and Y be connected (metric) spaces then the product space $X \times Y$ is connected

ASSIGNMENT SHEET - 1

1. Let F be the set of all functions from $[0, 1]$ to $[0, 1]$ itself. If $\text{card}(F) = f$ then

(a.) F is similar to $[0, 1]$

(b.) f is less than the cardinality of $[0, 1]$

[✓ (c.)] $f > c$ where cardnality of $[0,1]$ is c

(d.) F is countable

2. If $f : A \to B$ is a one-one map and A is countable. Then which is correct

(a.) B is countable.

(b.) B is uncountable

[✓ (c.)] There exists a subset of B which is countable

(d.) None of these.

3. Let A be an infinite set of disjoint open sub intervals of $(0, 1)$. Let B be the power set of A. Then-

(a.) Cardinality of A & B are equal

(b.) A similar to $(0, 1)$

[✓ (c.)] B similar to $(0, 1)$

(d.) A & B both are uncountable

4. Match the following list-I with list-II and choose the correct:

List-I

A. Countable

B. Uncountable

C. Empty set

List-II

1. Set of transcendental elements of $\mathbb{R}$ over $\mathbb{Q}$

2. Set of all function from $\mathbb{Z}_2 = \{0,1\}$ to N

3. $\{n \in \mathbb{N} : \sqrt{n+1} - \sqrt{n} \text{ is rational}\}$

Code:

	A	B	C
(a.)	2	3	1
(b.)	1	2	3
(c.)	3	2	1
[✓ (d.)]	2	1	3

5. Let A be the set of lines passing through the origin and slope is integral multiple of $\frac{\pi}{12}$. Then

(a.) A is similar to $\mathbb{R}$

[✓ (b.)] A is countably infinite

(c.) A is similar to the set of months in a year

(d.) A is similar to the power set of $\mathbb{R}$.

6. Consider the following statements:

1. The set of all finite subsets of the natural numbers is countable
2. The set of all polynomials with integer coefficients is countable

Choose the correct answer:

(a.) Only 1 is true

(b.) Only 2 is true

[✓ (c.)] Both 1 and 2 are true

(d.) Both are false

7. Let A & B are infinite sets. Let f is a map from A to B such that the collection of pre images of any non-empty subset of B is non empty. Then choose the incorrect

(a.) If A is countable then B is countable

(b.) Such map f is always onto

[✓ (c.)] A & B are similar

(d.) B may be countable even if A is not countable

8. If f be a function with domain A and range B then which of following is correct.

(a.) B countable $\Rightarrow A$ countable

[✓ (b.)] A countable $\Rightarrow B$ countable

(c.) A uncountable $\Rightarrow B$ uncountable

(d.) All of above

9. Which of the following is correct?

[✓ (a.)] The set of rational numbers in any interval of finite length is countable.

(b.) The set of irrational numbers in any interval of finite length is countable.

(c.) Every subset of uncountable set is uncountable.

(d.) All of above.

10. Read the following statements

1. S is a countable set
2. There exists a surjection of $\mathbb{N}$ onto S.
3. There exists an injection of S into $\mathbb{N}$

Codes:

(a.) 1 implies 2 & 3 but not conversely

(b.) 2 & 3 imply 1 but not conversely

(c.) 1 implies either of 2 & 3 but not both.

[✓ (d.) All the three statements are equivalent.]

11. Which set is countable

(a.) The set of all polynomials with real coefficients

(b.) The set of all subsets of a countably infinite set

(c.) The set $A-B$ where A is uncountable but B is countable

[✓ (d.) The set of all finite subsets of $\mathbb{N}$; (the set of natural number)]

12. If F be set of all function defined on $I_n=\{1,2,3,....,n\}; n\in\mathbf{N}$ with range $B\subseteq I$ (set of positive integer) Then

[✓ (a.) F is countable.]

(b.) F is uncountable

[✓ (c.) F is infinite]

[✓ (d.) F is countable if B is finite]

13. Select the correct statements

1. Every countable set is similar to $\mathbb{N}$
2. The set of all disjoint intervals is not similar to the set of real numbers
3. The power set of $\mathbb{N}$ is similar to the set of real numbers.

Codes:

(a.) 1 and 2 only

(b.) 2 and 3 only

[✓ (c.) 1 and 3 only]

(d.) All of these

14. Let for each pair of natural numbers (m,n), $P_{n,m}$ be the set of polynomials of degree n with integral coefficients s.t. $|a_0|+|a_1|+....+|a_n|=m$; where a_i's are coefficients then

(a.) $P_{m,n}$ is countably infinite

(b.) $P_{m,n}$ is finite for some m, n only

(c.) $P=\bigcup\{P_{mn}: m,n\in\mathbb{N}\}$ is set of all algebraic number.

[✓ (d.) P_{mn} is finite $\forall m,n\in\mathbb{N}$]

15. $S=\{a\in\mathbb{R}: 'a'$ is a recurring decimal number$\}$ $T=\{b\in\mathbb{R}: b=\sqrt{\frac{p}{q}}$ for some distinct primes p & $q\}$ then

(a.) S is countable but T is not

(b.) T is countable but S is not

[✓ (c.) S & T both are countable sets]

(d.) S & T are uncountable sets

16. Consider the following statements:

1. Every infinite set is equivalent to at least one of its proper subset.
2. If a set is equivalent to one of its proper subset then it is infinite set

Code:

(a.) 1 is correct and 2 is incorrect

(b.) 2 is correct and 1 is incorrect

[✓ (c.) Both are correct]

(d.) Both incorrect

17. Let $A_1, A_2, ..., A_n$ sets, where n be a fixed natural number. Consider following statements:

1. If $A=\bigcap_{i=1}^{n} A_i$ countably infinite, then there exist at least one A_i for $i=1,2,....n$ which is countable
2. If $A=A_1\times A_2\times ....\times A_n$ is countably infinite. Then each A_i for $i=1,2,....n$ is countable

Code:

(a.) 1 is correct and 2 is incorrect

[✓ (b.) 2 is correct and 1 is incorrect]

(c.) Both are correct

(d.) Neither 1 nor 2 are correct

18. How many statements is/are false?

1. Cardinality of $[0,1]\times[0,1]$ is the same as cardinality of $\mathbb{R}$
2. Cardinality of $\mathbb{R}$ is the same as the cardinality of irrationals
3. Cardinality of $\mathbb{R}$ is the same as the cardinality of $\mathbb{C}$

[✓ (a.) Zero]

(b.) One

(c.) Two

(d.) Three

19. Let A be an uncountable subset of $\mathbb{R}$ and B be a proper infinite subset of $\mathbb{N}$. Define $f: B \to A$ such that f is one-one, then

[✓ (a.)] A and $A-B$ are similar

[✓ (b.)] A and $A-f(B)$ are similar

[✓ (c.)] $\mathbb{R}$ and $\mathbb{R}-f(B)$ are similar

[✓ (d.)] All of the above are correct

20. Let P_n be the set of all polynomials of degree n with integral coefficients. The P_n is

(a.) A finite set with n^n elements

(b.) A finite set

[✓ (c.)] A countable set

(d.) An uncountable set

21. Consider the following statements.

1. S be set of all straight line in a plane each of which passes through at least two different rational co-ordinate
2. If $S=\{x : x \text{ be a rational point of } \mathbb{R}^2\}$ (a point $x=(x_1,x_2)\in\mathbb{R}^2$ called rational point if each $x_1, x_2 \in \mathbb{Q}$)

Then

[✓ (a.)] 1 and 2 both countable.

(b.) 1 countable and 2 uncountable

(c.) 2 countable and 1 uncountable

(d.) 1 and 2 both uncountable

22. Consider the following statements.

1. If every subset of a set is countable then set is countable.
2. If every proper subset of a set is countable then set is countable

Then

(a.) 1 correct 2 may or may not correct

(b.) 2 correct 1 may or may not correct

(c.) Both may or may not correct

[✓ (d.)] Both correct

23. If $F=\{A_1 A_2 .....\}$ is a countable collection of countable sets let $G=\{B_1 B_2 .....\}$ where $B_1 = A_1$ and for $n>1$,

$$B_n = A_n - \bigcup_{k=1}^{n-1} A_k .$$

Then form the following statements which is/are true

1. G is a collection of disjoint sets
2. $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k$
3. $\bigcup_{k=1}^{\infty} B_k$ is countable

Code:

(a.) Only 1

(b.) 1 and 3

(c.) 2 and 3

(d.) All three

24. Which of the following is/are true?

(a.) The set $\{e^x : x\in\mathbb{R}\}$ is a countable set

(b.) The set $\{\log x : x>0\}$ is a countable set

(c.) $\{\sin x : -\pi/2 \le x \le \pi/2\}$ is a countable set

[✓ (d.)] The set of all solutions of the equation $z^n = 1; n = 1, 2, \ldots$,where z is a complex number, is a countable set.

25. If $A=\{(x,y): y=e^x, x\in\mathbb{R}\}$ and $B=\{(x,y): y=e^{-x}, x\in\mathbb{R}\}$ then

(a.) $A\cap B=\phi$

[✓ (b.)] $A\cap B\neq\phi$

(c.) $A\cup B=\mathbb{R}$

(d.) None of these

26. Let X denote the two-point set $\{0,1\}$ and write $X_j=\{0,1\}$ for every $j=1,2,3,\ldots$ Let $Y=\prod_{j=1}^{\infty} X_j$. Which of the following is/are correct

(a.) Y is countable set

[✓ (b.)] cardinality $Y=$ cardinality of $[0,1]$

[✓ (c.)] $\bigcup_{n=1}^{\infty}\left(\prod_{j=1}^{n} X_j\right)$ is countable

[✓ (d.)] Y is uncountable

27. Which of the following subset of $\mathbb{R}^2$ is/are NOT countable set?

[✓ (a.)] $\{(a,b) \in \mathbb{R}^2 \mid a \le b\}$

[✓ (b.)] $\{(a,b) \in \mathbb{R}^2 \mid a+b \in \mathbb{Q}\}$

[✓ (c.)] $\{(a,b) \in \mathbb{R}^2 \mid ab \in \mathbb{Z}\}$

(d.) $\{(a,b) \in \mathbb{R}^2 \mid a,b \in \mathbb{Q}\}$

28. For each $J = 1,2,3...$ let A_j be a finite set containing at least two distinct element, then

[✓ (a.)] $\bigcup_{j=1}^{\infty} A_j$ is countable set

(b.) $\bigcup_{j=1}^{\infty} A_j$ is uncountable

[✓ (c.)] $\prod_{j=1}^{\infty} A_j$ is uncountable

(d.) $\prod_{j=1}^{\infty} A_j$ is countable

29. Consider the sets of sequences

$X = \{(x_n) : x_n \in \{0,1\} n \in \mathbb{N}\}$ and

$Y = \{(x_n) \in X : x_n = 1 \text{ for at most finitely many } n\}$

Then.

(a.) X is countable, Y is finite.

[✓ (b.)] X is uncountable, Y is countable.

(c.) X is countable, Y is countable.

(d.) X is uncountable, Y is uncountable

30. Let A be any set. Let $\mathbb{P}(A)$ be the power set of A, that is, the set of all subsets of $A; \mathbb{P}(A) = \{B : B \subseteq A\}$. Then which the following is/are true about the set $\mathbb{P}(A)$?

(a.) $\mathbb{P}(A) = \Phi$ for some A.

(b.) $\mathbb{P}(A)$ is a finite set for some A.

(c.) $\mathbb{P}(A)$ is a countable set for some A.

(d.) $\mathbb{P}(A)$ is a uncountable set for some A.

31. Which of the following sets of functions are uncountable? ($\mathbb{N}$ stands for the set of natural numbers.)

[✓ (a.)] $\{f \mid f : \mathbb{N} \to \{1,2\}\}$

(b.) $\{f \mid f : \{1,2\} \to \mathbb{N}\}$

(c.) $\{f \mid f : \{1,2\} \to \mathbb{N}, f(1) \le f(2)\}$

[✓ (d.)] $\{f \mid f : \mathbb{N} \to \{1,2\}, f(1) \le f(2)\}$

32. Let $A = (0, 1)$ & B is the sets of all disjoint open subintervals of A. then

(a.) A & B are similar

(b.) There is no onto map from A to B hence A & B are not similar.

(c.) There is no onto map from B to A hence they are not similar.

[✓ (d.)] There is no one-one map from A to B, hence they are not similar.

ASSIGNMENT SHEET - 2

1. Let G be the set of all irrational numbers. The interior and the closure of G are denoted by G° and $\bar{G}$, respectively. Then

(a.) $G^0=\phi$, $\bar{G}=G$

(b.) $G^0=\mathbb{R}$, $\bar{G}=\mathbb{R}$

[✓ (c.)] $G^0=\phi$, $\bar{G}=\mathbb{R}$

(d.) $G^0=G$, $\bar{G}=\mathbb{R}$

2. The set $\{x\in\mathbb{R}:x^2>x\}$ is same as

(a.) the interval (0,0)

(b.) the complement of the interval (0,1)

[✓ (c.)] the complement of the interval [0,1]

(d.) the interval [0,1].

3. The set of all boundary points of $\mathbb{Q}$ in $\mathbb{R}$ is

(a.) $\mathbb{R}$

(b.) $\mathbb{R}\setminus\mathbb{Q}$

[✓ (c.)] $\mathbb{Q}$

(d.) Empty set

4. Lim superior $\left\{\frac{(-1)^n}{2^n}:n=1,2,\dots\right\}$ is

[✓ (a.)] 0

(b.) $\frac{1}{4}$

(c.) 1

(d.) -1

5. If $Y=\left\{\frac{x}{1+|x|}\middle| x\in\mathbb{R}\right\}$, then the set of all limit points of Y is

(a.) $(-1,1)$

(b.) $(-1,1]$

(c.) $[0,1]$

[✓ (d.)] $[-1,1]$

6. The set $\left\{\frac{1}{n}\sin\frac{1}{n}:n\in\mathbb{N}\right\}$ has

[✓ (a.)] One limit point and it is 0

(b.) One limit point and it is 1

(c.) One limit point and it is -1

(d.) Three limit points and these are -1, 0 and 1.

7. The set $U=\left\{x\in\mathbb{R}\mid\sin x=\frac{1}{2}\right\}$ is

(a.) open

[✓ (b.)] closed

(c.) both open and closed

(d.) neither open nor closed

8. Let S be the set of all numbers of the form $\sum_{k=0}^{\infty}\frac{a_k}{3^k}$; $a_k=0$ or 2. Which of the following is true?

(a.) S is countable

(b.) S is dense in $[0,1]$

[✓ (c.)] S is closed in $[0,1]$

(d.) $S=[0,1]$

9. Consider the following subsets of $\mathbb{R}$: $E=\left\{\frac{n}{n+1}:n\in\mathbb{N}\right\}$, $F=\left\{\frac{1}{1-x}:0\le x<1\right\}$ Then

(a.) Both E and F are closed

(b.) E is closed and F is NOT closed

[✓ (c.)] E is NOT closed and F is closed

(d.) Neither E nor F is closed.

10. How many subsets A of {1,2,3,4} are there that $(A\cup\{1,2\})-(A\cap\{1,2\})$ has exactly one point?

(a.) 1

(b.) 2

(c.) 3

[✓ (d.)] 4

11. Let $A=\{m+n\sqrt{2}: m,n\in\mathbb{Z}\}$, where $\mathbb{Z}$ stands for the set of all integers

[✓ (a.)] A is dense in $\mathbb{R}$

(b.) A has only countable many limit point in $\mathbb{R}$

(c.) A has no limit point in $\mathbb{R}$

(d.) Only irrational numbers can be limit points of A

12. Let $X=\left\{\frac{1}{n}: n\in\mathbb{Z}, n\geq 1\right\}$ and let $\bar{X}$ be its closure. Then

[✓ (a.)] $\bar{X}\setminus X$ is a single point

(b.) $\bar{X}\setminus X$ is open in $\mathbb{R}$

(c.) $\bar{X}\setminus X$ is infinite but not open in $\mathbb{R}$

(d.) $\bar{X}\setminus X=\phi$

13. Which of the following sets satisfy the condition that for every positive integer n there is some a in A such that $a<n$?

[✓ (a.)] $A=\{-1,5\}$

(b.) A is empty set

(c.) $A=\mathbb{N}$

(d.) $A=\{x\in\mathbb{R}\mid x>10\}$

14. Let S be an infinite subset of $\mathbb{R}$ such that $S\cap\mathbb{Q}=\phi$. Which of the following statements is true?

(a.) S must have a limit point which belongs to $\mathbb{Q}$

(b.) S must have a limit point which belongs to $\mathbb{R}\setminus\mathbb{Q}$

(c.) S cannot be a closed set in $\mathbb{R}$

[✓ (d.)] $\mathbb{R}/S$ must have a limit point which belongs to S.

15. Let $A\neq\mathbb{R}$ be a dense subset of $\mathbb{R}$. If $U\subseteq\mathbb{R}$ is a non-empty open subset then

[✓ (a.)] $U\subseteq\overline{A\cap U}$

(b.) $\overline{A\cap U}=\phi$

(c.) $\overline{A\cap U}\subseteq U$

(d.) $\overline{A\cap U}=A\cap\bar{U}$

16. If $S=\left\{\frac{1}{2m}+\frac{1}{3n};\ m,n\in N\right\}$ then the derived sets S' and S'' has the property

(a.) S' & S'' both are finite

(b.) S' is finite & S'' is empty

[✓ (c.)] S' is infinite & S'' is singleton

(d.) S' & S'' both are infinite

17. Let $A=\left\{1+\frac{1}{n}: n\in N\right\}$ and $B=\left\{1-\frac{1}{n}: n\in N\right\}$ then derived set of $(A\cup B)$

(a.) Has cardinality 2

[✓ (b.)] Has cardinality 1

(c.) Is empty set

(d.) Is countably infinite set

18. Let $S=\{x\sqrt{2}: x\in\mathbb{Q}\}$ then the derived set of S is

(a.) Empty set

(b.) Singleton set

(c.) Equivalent to the set of natural numbers

[✓ (d.)] Uncountable set

19. In $\mathbb{R}$, let $F_n=\left(\frac{-1}{n},\frac{1}{n}\right), n=1,2,3,.........$ Then $\bigcap_{n=1}^{\infty}F_n$ is

(a.) Empty

(b.) Open but not closed

[✓ (c.)] Closed but not open

(d.) Both open and closed

20. Which of the following statement(s) is(are) TRUE?

[✓ (a.)] There exists a connected set in $\mathbb{R}$ which is not compact

[✓ (b.)] Arbitrary union of closed intervals in $\mathbb{R}$ need not be compact

(c.) Arbitrary union of closed intervals in $\mathbb{R}$ is always closed

(d.) Every bounded infinite subset V of $\mathbb{R}$ has a limit point in V itself

21. Let A be a non empty subset of $\mathbb{R}$ having a supremum. Let B=$\{x \in \mathbb{R} : (x+1)/2 \in A\}$ then

[✓ (a.) sup B = 2 sup A- 1]

(b.) Sup B =(sup $A+1)/2$

(c.) Sup B = sup A

(d.) B does nothave supremum

22. Let A be any subset of $\mathbb{R}$ such that $A \cap A' = \phi$ where A' is set of all limit point of A. Then,

(a.) A is uncountable

(b.) A is finite

[✓ (c.) A is countable]

(d.) None of these

23. The supremum of the set

$$S = \left\{1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{2^2} \ldots, 1+\frac{1}{2}+\frac{1}{2^2} \cdots \frac{1}{2^{n-1}}\right\}$$

(a.) 1

[✓ (b.) 2]

(c.) 3

(d.) $\frac{1}{2}$

24. Let $A = \left\{x : x = \frac{4n+3}{n}; n \in \mathbb{N}\right\}$ then sup A & inf A are respectively

[✓ (a.) 7 & 4]

(b.) 0 & 4

(c.) 3 & 4

(d.) None of these

25. Select the incorrect statement about the set $A = \left\{1, 1+\frac{1}{3}, 1+\frac{1}{3}+\frac{1}{3^2}, \ldots, 1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3} \ldots + \frac{1}{3^{n-1}} \ldots\right\}$

(a.) The set A is bounded.

(b.) The inf A is 1

[✓ (c.) The set A has its largest elements]

(d.) None of these.

26. $S = \left[(-1)^n\left(\frac{1}{4}-\frac{4}{n}\right) : n \in \mathbb{N}\right]$ Then

(a.) $Sup\ S = \frac{1}{4}$ and $\inf S = \frac{1}{4}$

[✓ (b.) $Sup\ S = \frac{15}{4}$ and $\inf S = \frac{-7}{4}$]

(c.) $Sup\ S = \frac{15}{4}$ and $\inf S = -\frac{1}{4}$

(d.) None of these

27. S be subset of $\mathbb{R}$ and $\inf S = \sup S$, Then

(a.) S empty

[✓ (b.) S singleton]

(c.) S finite but may not be singleton

(d.) Can't say

28. $S = \left\{\left(1-\frac{1}{n}\right)\sin\frac{n\pi}{2} : n \in \mathbb{N}\right\}$. Then correct statement.

(a.) S has smallest element but not greatest element

[✓ (b.) S has greatest element but not smallest element]

(c.) S has both greatest & least element

(d.) S bounded but have not smallest and greatest element both

29. Which of the following is not a *nbd* of each of its point?

(a.) Set $\mathbb{Q}$ of rational numbers

(b.) Set $\mathbb{Q}^c$ of irrational number

(c.) Set $\mathbb{Z}$ of integers

[✓ (d.) None of these]

30. Let S be an uncountable set and T be a set of those real number x s.t. $(x-\delta, x+\delta) \cap S$ is uncountable then which of the statements is/are correct

(a.) T is countable

[✓ (b.) $S-T$ is countable]

[✓ (c.) $S \cap T$ is uncountable]

(d.) All are correct

31. Let E_i are subsets of $\mathbb{R}$ such that. $E_i \cap E_i' = \phi$. Then select the correct statement:

(a.) Each E_i is countable

(b.) $\mathbb{R} - \cup E_i$ is uncountable for $i = 1$ to n

(c.) $\mathbb{R} - \cup E_i$ is uncountable $\forall i \in \mathbb{N}$

(d.) None of these.

32. Choose the incorrect statement:

(a.) $S \subset T \Leftrightarrow S' \subset T'$

(b.) $x \in S' \Leftrightarrow x \in (S \cup \{x\})'$ where S is any subset of $\mathbb{R}$

(c.) If $S = \left\{\frac{1}{2m} + \frac{1}{2n}; m, n \in \mathbb{N}\right\}$, then $(S')' = \{0\}$

(d.) $(1,3) = \cup\left[1+\frac{1}{n}, 3-\frac{1}{n}\right] \forall n \in \mathbb{N}$

33. Consider the following statements and choose correct

(a.) Every infinite and bounded set must have a limit point

(b.) Any finite set cannot have a limit point

(c.) Any infinite but unbounded set can't have a limit point

(d.) None of these

34. Let A be a closed subset of $\mathbb{R}$, $A \neq \emptyset, \mathbb{R}$.

Then A is

(a.) the closure of the interior of A.

(b.) a countable set.

(c.) a compact set.

(d.) not open.

35. Let $E \subset \mathbb{R}, E \neq \phi$. Let (1), (2) and (3) denote the following conditions:

E is infinite

E is bounded

E is closed

(a.) 1 is necessary for E to have a limit point

(b.) 1 and 2 together are sufficient for E to have a limit point

(c.) 1 and 3 together are sufficient for E to have a limit point

(d.) 3 is sufficient for every limit point of E to belong to E

36. Which of the following subsets of $\mathbb{R}$ is closed?

(a.) $[0, 1] \cup [2, 3] \cup [4, 5]$

(b.) $[0, 1[$

(c.) $(1, \infty)$

(d.) The set of rational numbers in $[0, 1]$

37. Which set is/are not open?

(a.) $\bigcap_{n=1}^{\infty} G_n$, where $G_n =]-\frac{1}{n}, \frac{1}{n}[$

(b.) The set $\mathbb{Q}$ of rational number

(c.) The union of arbitrary family of open set

(d.) All of above

38. Let $s \in (0,1)$. Then decide which of the following are true.

(a.) $\forall m \in \mathbb{N}, \ni n \in \mathbb{N}\, s.t.\ s > m/n$

(b.) $\forall m \in \mathbb{N}, \ni n \in \mathbb{N}\, s.t.\ s < m/n$

(c.) $\forall m \in \mathbb{N}, \ni n \in \mathbb{N}\, s.t.\ s = m/n$

(d.) $\forall m \in \mathbb{N}, \ni n \in \mathbb{N}\, s.t.\ s = m + n$

39. Let G and H be nonempty subsets of $\mathbb{R}$ where G is connected and $G \cup H$ is not connected. Which one of the following statements is true for all such G and H?

(a.) If $G \cap H = \varnothing$, then H is connected

(b.) If $G \cap H = \varnothing$, then H is not connected

(c.) If $G \cap H \neq \varnothing$, then H is connected

(d.) If $G \cap H \neq \varnothing$, then H is not connected

40. Let S be a nonempty subset of $\mathbb{R}$. If it is a finite union of disjoint bounded intervals, then which one of the following is true?

(a.) If S is not compact, then $\sup S \notin S$ and $\inf S \notin S$

(b.) Even if $\sup S \in S$ and $\inf S \in S$, S need not be compact

(c.) If $\sup S \in S$ and $\inf S \in S$, then S is compact

(d.) Even if S is compact, it is not necessary that $\sup S \in S$ and $\inf S \in S$

ASSIGNMENT SHEET - 3

1. The sequence $a_n = \sqrt{A^2n^2 + n+1} - n$ is
   - (a.) convergent for all real values of A
   - (b.) divergent for all real values of A
   - (c.) convergent for exactly one real value of A
   - [✓ (d.)] Convergent for exactly two real values of A

2. The least upper bound of the sequence $\left\langle 1-\frac{1}{n}\right\rangle$ is
   - (a.) 0
   - (b.) -1
   - [✓ (c.)] 1
   - (d.) None of these

3. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers defined as $a_1 = 1$ and for $n \geq 1$, $a_{n+1} = a_n + (-1)^n 2^{-n}$; $b_n = \frac{2a_{n+1} - a_n}{a_n}$. Then
   - (a.) $\{a_n\}$ converges to zero and $\{b_n\}$ is a Cauchy sequence
   - [✓ (b.)] $\{a_n\}$ converges to a non-zero number and $\{b_n\}$ is a Cauchy sequence
   - (c.) $\{a_n\}$ converges to zero and $\{b_n\}$ is not a convergent sequence
   - (d.) $\{a_n\}$ converges to a non-zero number and $\{b_n\}$ is not a convergent sequence

4. Which of the following sequence $\langle a_n \rangle$ is monotonic:
   - (a.) $\langle a_n \rangle = 1 - \frac{(-1)^n}{n}$
   - (b.) $a_{n+2} = \frac{1}{2}(a_n + a_{n+1})$, where $a_1 < a_2$ and a_1, a_2 are given
   - [✓ (c.)] $a_{n+1} = \sqrt{\frac{ab^2 + a_n^2}{a+1}} \cancel{+} n$, where $a > 0$, $0 < a_1 < b$ and $a_1 = a$
   - (d.) None of the above

5. Define a sequence s_n by $s_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k}}$ Then the limit of s_n as n tends to infinity
   - (a.) is 0
   - [✓ (b.)] is 1
   - (c.) is ∞
   - (d.) doesn't exist

6. Let $\{x_n\}$ be a real sequence. If sequence of even terms of $\{x_n\}$ converges to 1 and sequence of odd terms converges to -1. Then the sequence $\{x_n\}$ will
   - (a.) Converge to 0
   - (b.) Converge to 1
   - (c.) Converge to -1
   - [✓ (d.)] None of these

7. Let $S_n = \sum_{k=2}^{n} \frac{(-1)^k}{k \ln k}$. Then the sequence $\{S_k\}$
   - [✓ (a.)] converges to a finite number
   - (b.) diverges to ∞
   - (c.) diverges to $-\infty$
   - (d.) oscillates

8. Consider the interval $(1-,1)$ and a sequence $\{a_n\}_{n=1}^{\infty}$ of elements in it. Then,
   - (a.) Every limit point of $\{a_n\}$ is in $(-1,1)$
   - [✓ (b.)] Every limit point of $\{a_n\}$ is in $[-1,1]$
   - (c.) The limit points of $\{a_n\}$ can only be in $\{-1,0,1\}$
   - (d.) The limit points of $\{a_n\}$ cannot be in $\{-1,0,1\}$

9. Consider the sequence
   4,0,4.1,0,4.11,0, 4.111,0,....
   Then
   - (a.) converges to $4\frac{1}{9}$
   - (b.) is divergent
   - (c.) is unbounded
   - (d.) is not convergent and has supremum $= 4\frac{1}{9}$

10. Let $x_n = 2^{2n}\left(1-\cos\left(\frac{1}{2^n}\right)\right)$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$

(a.) does NOT converge

(b.) converges to 0

[✓ (c.) converges to $\frac{1}{2}$]

(d.) converges to $\frac{1}{4}$

11. Let $p(x)$ be a polynomial in the real variable x of degree 5. Then $\lim_{n\to\infty}\frac{p(n)}{2^n}$ is

(a.) 5

(b.) 1

[✓ (c.) 0]

(d.) ∞

12. If $0 < c < d$, then the sequence $a_n = (c^n + d^n)^{1/n}$

(a.) is bounded and monotonically decreasing

(b.) is bounded and monotonially increasing

(c.) is monotonically increasing, but unbounded for $1 < c < d$

(d.) is monotonically decreasing, but unbounded for $1 < c < d$

13. The limit superior and the limit inferior of the following sequence $\left\langle (-1)^n\left(1+\frac{1}{n}\right)\right\rangle$ are

[✓ (a.) $1, -1$]

(b.) $2, -1$

(c.) $2, 1$

(d.) $1, 1$

14. Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of real numbers such that $b_n = a_{2n}$ and $c_n = a_{2n+1}$. Then $\{a_n\}$ is convergent

(a.) implies $\{b_n\}$ is convergent but $\{c_n\}$ need not be convergent

(b.) implies $\{c_n\}$ is convergent but $\{b_n\}$ need not be convergent

(c.) implies both $\{b_n\}$ and $\{c_n\}$ are convergent

[✓ (d.) if both $\{b_n\}$ and $\{c_n\}$ are convergent]

15. For each positive integer n, let a_n be the number of points of intersection of the graph $y = \sin x$ with the line $y = \frac{x}{n}$. The sequence $\{a_n\}$ is

(a.) decreasing

(b.) constant

(c.) converging to zero

[✓ (d.) diverging to infinity]

16. The value of $\lim_{x\to\infty}\left(\frac{a_1^{1/x}+a_2^{1/x}+.......+a_n^{1/x}}{n}\right)^{nx}$, $a_i > 0, \forall i$, is

(a.) $\frac{a_1+a_2......+a_n}{n}$

(b.) $e^{a_1+a_2+...+a_n}$

(c.) $a_1+a_2+...+a_n$

[✓ (d.) $a_1a_2, ..., a_n$]

17. Let $\langle x_n\rangle$ be a convergent sequence and $\langle y_n\rangle$ be a monotonic sequence. Then $\langle x_n \cdot y_n\rangle$

(a.) Always converges

(b.) Converges if $\langle x_n\rangle$ is monotonic

(c.) Converges if $\langle x_n y_n\rangle$ is monotonic

[✓ (d.) Converges if $\langle y_n\rangle$ is bounded]

18. Consider the sequence $\{l_n\}\ n \in \mathbb{N}$ with $l_n = \frac{1}{n+1}+...+\frac{1}{2n}$. This sequence

[✓ (a.) is increasing and bounded]

(b.) increases to ∞

(c.) decreases to 0

(d.) decreases to a positive number

19. The sequence $\{S_n\}$, where $S_n = 1+\frac{1}{4}+\frac{1}{7}+...+\frac{1}{3n-2}$

(a.) A Cauchy sequence

(b.) A convergent sequence

[✓ (c.) Cannot converge]

(d.) None of these

20. The value of $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+kn}}$ is

[✓ (a.) $2(\sqrt{2}-1)$]
(b.) $2\sqrt{2}-1$
(c.) $2-\sqrt{2}$
(d.) $\frac{1}{2}(\sqrt{2}-1)$

21. Consider the sequence $\{a_n\}$ of real numbers where $a_1>1$ and $a_{n+1}=2-\frac{1}{a_n}, n\geq 1$. Then the sequence $\{a_n\}$ is

(a.) bounded but not monotone.
(b.) not bounded but monotone
[✓ (c.) both bounded and monotone.]
(d.) neither bounded nor monotone.

22. Let $\langle a_n\rangle$ be a sequence of real numbers define b_n and c_n as

$b_n = n^{th}$ rational terms in $\langle a_n\rangle$, $c_n = n^{th}$ irrational terms in $\langle a_n\rangle$.

Then

(a.) $\langle b_n\rangle$ and $\langle c_n\rangle$ both are subsequence, of $\langle a_n\rangle$
(b.) Both $\langle b_n\rangle$ and $\langle c_n\rangle$ may fail to be subsequence, of $\langle a_n\rangle$
[✓ (c.) At least one of $\langle b_n\rangle$ and $\langle c_n\rangle$ is always a subsequence of $\langle a_n\rangle$]
(d.) None of the above

23. The largest term in sequence $x_n=\frac{1000^n}{n!}, n=1,2,3,.....$

[✓ (a.) is x_{999}]
(b.) is x_{1001}
(c.) is x_1
(d.) does not exist

24. Let the sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers converges to a non zero real number a and let $y_n=a-x_n$. Then $\max_{n\in\mathbb{N}}\{x_n,y_n\}$ converges to

(a.) a always
(b.) 0 always
[✓ (c.) $\max\{a,0\}$]
(d.) $\min\{a,0\}$

25. Let $\{x_n\}$ be an unbounded sequence of non-zero real numbers. Then,

(a.) $\{x_n\}$ must have a convergent subsequence
(b.) $\{x_n\}$ cannot have a convergent subsequence
[✓ (c.) $\left\{\frac{1}{x_n}\right\}$ must have a convergent subsequence]
(d.) $\left\{\frac{1}{x_n}\right\}$ cannot have a convergent subsequence.

26. Let $\{a_n\}$ be a sequence of real numbers. Let $b_n=a_n+a_{n+1}$, for $n=1,2,...$ Which of the following is always true:

[✓ (a.) If $\{b_n\}$ converges, then $\{a_n\}$ converges.]
(b.) If $\{a_n\}$ converges, then $\{b_n\}$ diverges.
(c.) If $\{a_n\}$ converges, then $\{b_n\}$ converges.
(d.) $\{a_n\}$ is a subsequence of $\{b_n\}$.

27. $x_{n+1}=\frac{-3}{4}x_n$, $x_0=1$. The sequence $\{x_n\}$

(a.) diverges
(b.) x_n is monotonically increasing and converges to 0
(c.) x_n is monotonically decreasing and converges to 0
(d.) none of the above

28. Let $\{x_n\}$ be sequence of real numbers such that $\lim_{n\to\infty}(x_{n+1}-x_n)=c$, where c is a positive real number. Then the sequence $\left\{\frac{x_n}{n}\right\}$.

(a.) is NOT bounded
(b.) is bounded but NOT convergent
(c.) converges to c
(d.) converges to 0

29. If $\langle a_n\rangle$ is a convergent sequence then $\lim_{n\to\infty}n(a_{n+1}-a_n)=$

(a.) ∞
(b.) 1
(c.) e
[✓ (d.) 0]

30. Suppose $a>0$. Consider the sequence $a_n = n\{\sqrt[n]{ea} - \sqrt[n]{a}\}$, $n \geq 1$. Then

(a.) $\lim\limits_{n\to\infty} a_n$ does not exist

(b.) $\lim\limits_{n\to\infty} a_n = e$

(c.) $\lim\limits_{n\to\infty} a_n = 0$

(d.) None of the above.

31. $\lim\limits_{n\to\infty}\left[\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \cdots\cdots\cdots \left(\frac{n+1}{n}\right)^n\right]^{1/n} =$

(a.) 1

(b.) –1

(c.) $-e$

(d.) e

32. Given a_1, a_2 & $\forall n \in \mathbf{N}$,

$a_n = \left(1-\frac{1}{n}\right)a_{n-1} + \frac{1}{n}a_{n-2}$. Then $\lim\limits_{n\to\infty} a_n =$

(a.) $2(a_2 - a_1) + a_1 e^{-1}$

(b.) $2(a_1 - a_2)e^{-1} + a_2$

(c.) $2(a_1 - a_2)e^{-1} + a_1$

(d.) $2(a_2 - a_1)e^{-1} + a_1$

33. Which of the following is/are correct?

(a.) $n\log\left(1+\frac{1}{n+1}\right) \to 1$ as $n \to \infty$

(b.) $(n+1)\log\left(1+\frac{1}{n+1}\right) \to 1$ as $n \to \infty$

(c.) $n^2\log\left(1+\frac{1}{n}\right) \to 1$ as $n \to \infty$

(d.) $n\log\left(1+\frac{1}{n^2}\right) \to 1$ as $n \to \infty$

34. If $\{x_n\}$ and $\{y_n\}$ are sequences of real numbers, which of the following is/are true?

(a.) $\limsup\limits_n (x_n + y_n) \leq \limsup\limits_n x_n + \limsup\limits_n y_n$

(b.) $\limsup\limits_n (x_n + y_n) \geq \limsup\limits_n x_n + \limsup\limits_n y_n$

(c.) $\liminf\limits_n (x_n + y_n) \leq \liminf\limits_n x_n + \liminf\limits_n y_n$

(d.) $\liminf\limits_n (x_n + y_n) \geq \liminf\limits_n x_n + \liminf\limits_n y_n$

35. If $\langle a_n \rangle$ is a sequence of real numbers such that for some $l \in R$, define a set $K = \{n \in N : a_n \notin (l-\varepsilon, l+\varepsilon), \forall\, \varepsilon > 0\}$ if K is bounded, then choose the incorrect

(a.) l is a limit point of $\langle a_n \rangle$ but $\langle a_n \rangle$ may fail to be convergent

(b.) $\langle a_n \rangle$ is monotonic.

(c.) $\langle a_n \rangle$ is a Cauchy sequence

(d.) $\langle a_n \rangle$ is bounded but need not be convergent

36. Let $\langle a_n \rangle$ and $\langle b_n \rangle$ are two sequences. Then choose the incorrect

(a.) If $\langle a_n + b_n \rangle$ and $\langle a_n \cdot b_n \rangle$ are both convergent. Then $\langle a_n \rangle$ and $\langle b_n \rangle$ are convergent

(b.) If $\langle a_n - b_n \rangle$ and $\left\langle \frac{a_n}{b_n} \right\rangle$ are both convergent. Then $\langle a_n \rangle$ and $\langle b_n \rangle$ are convergent

(c.) If $\langle a_n + b_n \rangle$ and $\langle a_n - b_n \rangle$ are both convergent. Then $\langle a_n \rangle$ and $\langle b_n \rangle$ are convergent

(d.) If $\langle a_n + b_n \rangle$ and $\langle a_n - b_n \rangle$ are both convergent. Then $< a_n . b_n >$ convergent

37. Which of the following are null sequences

(a.) $\left\{\frac{n}{\alpha^n}\right\} : \alpha > 1$

(b.) $\left\{\frac{\log n}{n}\right\}$

(c.) $\{\alpha^n\} : |\alpha| < 1$

(d.) $\left(1+\frac{3}{n}\right)^n : n \in \mathbf{N}$

38. Which of following is/are Incorrect statement

(a.) Every sequence contains a monotonic subsequence

(b.) Every limit point of the sequence is the limit of the range set of sequence

(c.) If a sequence has a unique limit point then it must be limit of the sequence

(d.) The set of disjoint open intervals in **R** is an uncountable set.

39. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers such that $x_n \le y_n \le x_{n+2}$, $n = 1, 2, 3, \ldots$

(a.) $\{y_n\}$ is a bounded sequence.

(b.) $\{x_n\}$ is an increasing sequence.

[✓ (c.)] $\{x_n\}$ and $\{y_n\}$ converge together.

(d.) $\{y_n\}$ is an increasing sequence.

40. Let $\langle a_n \rangle$ be a sequence of real numbers defined as $a_n = \begin{cases} -1-\dfrac{1}{n} & n \text{ is even} \\ 1+\dfrac{1}{n} & n \text{ is odd} \end{cases}$

Then which of the following is correct:

[✓ (a.)] $\langle a_n \rangle$ is osciilatory

(b.) If lim inf $\langle a_n \rangle = a$ and lim sup $\langle a_n \rangle = b$ then $a_n \notin (a, b)$ for $\forall\, n \in \mathrm{N}$

(c.) $S = \{\lim a_{n_k} : \langle a_{n_k} \rangle$ is a convergent subsequent of $\langle a_n \rangle\}$ is finite

(d.) None of these

41. Which of the following is not possible about $\langle a_n \rangle$

[✓ (a.)] Σa_n is convergent but $\langle a_n \rangle$ is non convergent

(b.) $\langle |a_n| \rangle$ is convergent but $\langle a_n \rangle$ is non convergent

[✓ (c.)] $\langle a_n \rangle$ is convergent but $\langle |a_n| \rangle$ is non convergent

[✓ (d.)] None of these

42. If $\{a_n\}$ is a sequence of real numbers with $\limsup_{n\to\infty} a_n = 1$, then

(a.) Set of values taken by $\{a_n\}$ is not a finite set

(b.) $\{a_n\}$ is convergent

(c.) $\{a_n\}$ is bounded

[✓ (d.)] $\{a_n\}$ cannot diverge to $+\infty$

43. Let $x_1 = 0, x_2 = 1$, and for $n \ge 3$, define $x_n = \dfrac{x_{n-1} + x_{n-2}}{2}$. Which of the following is/are true?

(a.) $\{x_n\}$ is a monotone sequence.

(b.) $\lim_{n\to\infty} x_n = \dfrac{1}{2}$.

[✓ (c.)] $\{x_n\}$ is a Cauchy sequence.

[✓ (d.)] $\lim_{n\to\infty} x_n = \dfrac{2}{3}$

44. $\lim_{n\to\infty} \dfrac{1}{\sqrt{n}}\left(\dfrac{1}{\sqrt{2}+\sqrt{4}} + \dfrac{1}{\sqrt{4}+\sqrt{6}} + \ldots \dfrac{1}{\sqrt{2n}+\sqrt{2n+2}}\right)$ is

(a.) $\sqrt{2}$

[✓ (b.)] $\dfrac{1}{\sqrt{2}}$

(c.) $\sqrt{2}+1$

(d.) $\dfrac{1}{\sqrt{2}+1}$

45. Let $S_n = \sum_{k=1}^{n} \dfrac{1}{k}$. Which of the following is true?

[✓ (a.)] $S_{2^n} \ge \dfrac{n}{2}$ for every $n \ge 1$.

(b.) S_n is a bounded sequence.

(c.) $\left|S_{2^n} - S_{2^{n-1}}\right| \to 0$ as $n \to \infty$.

(d.) $\dfrac{S_n}{n} \to 1$ as $n \to \infty$.

ASSIGNMENT SHEET-4

1. If $b_n = \begin{cases} \dfrac{1}{\sqrt{n}} & \text{if } n \text{ is odd} \\ \dfrac{1}{n} & \text{if } n \text{ is even} \end{cases}$

Then

(a.) Both $\langle b_n \rangle$ and $\sum b_n$ are convergent

(b.) Both $\langle b_n \rangle$ and $\sum b_n$ are divergent

[✓ (c.)] $\langle b_n \rangle$ is convergent but $\sum b_n$ is not

(d.) $\sum b_n$ is convergent but $\langle a_n \rangle$ is convergent

2. Consider the sequence $\left\{\dfrac{(-1)^n}{n}\right\}$ and the series $\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}$. Then

(a.) the sequence converges but not the series.

(b.) the series converges but not the sequence.

(c.) neither the series not the sequence converges.

[✓ (d.)] both the series and the sequence converge.

3. Which of the following conditions does NOT ensure the convergence of a real sequence $\{a_n\}$?

[✓ (a.)] $|a_n - a_{n+1}| \to 0$ as $n \to \infty$

(b.) $\sum_{n=1}^{\infty}|a_n - a_{n+1}|$ is convergent

(c.) $\sum_{n=1}^{\infty} na_n$ is convergent

(d.) The sequence $\{a_{2n}\}, \{a_{2n+1}\}$ and $\{a_{3n}\}$ are convergent.

4. The series $\sum\left\{\dfrac{1}{2^{n-1}} + (-1)^{n-1}\right\}$ is

(a.) Convergent

(b.) divergent

(c.) oscillating infinitely

[✓ (d.)] oscillating finitely

5. The sum of the series $\dfrac{1}{1!} + \dfrac{1+2}{2!} + \dfrac{1+2+3}{3!} + \ldots$ equals

(a.) e

(b.) $\dfrac{e}{2}$

[✓ (c.)] $\dfrac{3e}{2}$

(d.) $1 + \dfrac{e}{2}$

6. Let $\{a_n\}$ be an increasing sequence of positive real numbers such that the series $\sum_{k=1}^{\infty} a_k$ is divergent. Let $S_n = \sum_{k=1}^{n} a_k$ for $n = 1, 2, \ldots$ and $t_n = \sum_{k=2}^{n} \dfrac{a_k}{S_{k-1}S_k}$ for $n - 2, 3, \ldots$

Then $\lim\limits_{n\to\infty} t_n$ is equal to

[✓ (a.)] $\dfrac{1}{a_1}$

(b.) 0

(c.) $\dfrac{1}{(a_1 + a_2)}$

(d.) $a_1 + a_2$

7. In each of the following cases, which of the series is absolutely convergent.

(a.) $\sum_{n=1}^{\infty}(-1)^n \dfrac{1}{2n+3}$

(b.) $\sum_{n=1}^{\infty}(-1)^n \dfrac{n}{n+2}$

[✓ (c.)] $\sum_{n=1}^{\infty}(-1)^n \dfrac{n\log n}{e^n}$

(d.) None of these

8. Let $\{x_n\}$ be a sequence of real numbers so that $\sum_{n=1}^{\infty}|x_n - x| = c$, with c finite. Then

(a.) $\{x_n\}$ may not be bounded

[✓ (b.)] $\{x_n\}$ must converge to x

(c.) $\{x_n\}$ must converge to $x + c$

(d.) $\{x_n\}$ is bounded but not necessarily convergent.

9. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series, where $a_n = \frac{(-1)^n n}{2^n}, b_n = \frac{(-1)^n}{\log(n+1)}$ for all $n = \mathbb{N}$. Then

(a.) both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent

[✓ (b.) $\sum_{n=1}^{\infty} a_n$ is absolutely convergent but $\sum_{n=1}^{\infty} b_n$ is conditionally convergent]

(c.) $\sum_{n=1}^{\infty} a_n$ is conditionally convergent but $\sum_{n=1}^{\infty} b_n$ is absolutely convergent

(d.) both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_n$ are conditionally convergent

10. We are given a convergent series $\sum_{n=1}^{\infty} a_n$, where $a_n \geq 0$ for each n. Which of the following correctly describes the behavior of the series $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n^p}$, $1 \leq p \leq 2$?

(a.) Diverges when $p = 1$, but converges for $p \in (1, 2]$

[✓ (b.) Converges for every $p \in [1, 2]$]

(c.) Diverges when $p \in \left[1, \frac{5}{4}\right]$, but converges for $p \in \left[\frac{5}{4}, 2\right]$

(d.) Diverges for every $p \in [1, 2]$

11. Consider the series $\sum \frac{1}{\sqrt{n}}$ and $\sum \frac{1}{n^{3/2}}$. Then

(a.) both the series converge to the same value.

(b.) both the series converge to different values.

(c.) both the series are divergent.

(d.) first series is divergent and second series is convergent.

12. If the terms of this oscillating series are grouped pairwise $3 - \frac{2^2+1}{2} + \frac{2^3+1}{2^2} - \frac{2^4+1}{2^3} + \ldots$ then the resulting series becomes

[✓ (a.) Convergent]

(b.) Divergent

(c.) Oscillate finitely

(d.) Oscillate infinitely

13. Which of the following series is divergent?

(a.) $\sum_{n=1}^{\infty} \sqrt{n} \sin\frac{1}{n}$

(b.) $\sum_{n=1}^{\infty} \sqrt{n} \sin^{-1}\left(\frac{1}{n^2}\right)$

(c.) $\sum_{n=1}^{\infty} n \tan\left(\frac{1}{n^3}\right)$

(d.) $\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n^2}\right)$

14. The largest interval in which the series $\sum_{n=1}^{\infty} x^n$ converges

[✓ (a.) For x with $-1 < x < 1$]

(b.) For x with $-\frac{1}{2} < x < \frac{1}{2}$

(c.) For x with $-2 < x < 2$

(d.) For x with $-\frac{1}{2} \leq x \leq \frac{1}{2}$

15. The series

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \ldots \text{ is}$$

[✓ (a.) convergent]

(b.) divergent

(c.) oscillating finitely

(d.) oscillating infinitely

16. The series $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \ldots$ is convergent if

(a.) $0 < x < \frac{1}{e}$

(b.) $x > \frac{1}{e}$

(c.) $\frac{2}{e} < x < \frac{3}{e}$

(d.) $\frac{3}{e} < x < \frac{4}{e}$

17. Consider the following statements:

1. If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms, then the convergence of $\sum_{n=1}^{\infty}(-1)^n u_n$ implies the convergence of $\sum_{n=1}^{\infty} u_n$.
2. If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms then the convergence of $\sum_{n=1}^{\infty} u_n$ implies the convergence of $\sum_{n=1}^{\infty}(-1)^n u_n$
3. The convergence of $\sum_{n=1}^{\infty} u_n \quad (u_n > 0)$ implies the convergence of $\sum_{n=1}^{\infty} u_n^2$.

Select the correct answer using the codes given below

(a.) 2 and 3
(b.) 1 and 3
(c.) 1 and 2
(d.) 1, 2 and 3

18. Let the series $\sum_{n=1}^{\infty} u_n$ has bounded partial series, then the series $\sum_{n=1}^{\infty} u_n v_n$ is convergent if the sequence $\{v_n\}$ is

(a.) Monotonic and bounded
(b.) Monotonic and converges to zero
(c.) Bounded and converges to zero
(d.) Bounded and monotonic which converges to zero.

19. If p is a real number, then the series $\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+.....$ to ∞ is convergent for

(a.) $p>0$
(b.) $p>1$
(c.) $p>2$
(d.) $p<1$

20. The series $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+.....$ to ∞ is

(a.) Convergent
(b.) Divergent
(c.) Oscillating
(d.) Converges conditionally

21. $\sum_{n=1}^{\infty}\frac{1}{\left(n^2+1\right)^k}$ is convergent for

(a.) $k<\frac{1}{2}$
(b.) $k>\frac{1}{2}$
(c.) $k=\frac{1}{2}$
(d.) All values of k

22. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{3n+k}$ is

(a.) $\log\frac{4}{3}$
(b.) $\log\frac{3}{4}$
(c.) $\log\frac{3}{2}$
(d.) $\log\frac{5}{4}$

23. $\sum \frac{n^n}{(n+a)^n}a \quad : a>0$

(a.) Converges for all $a>1$
(b.) Converges for $a>0$
(c.) Converges for $a\geq 0$
(d.) Div for all a

24. Which of the following series are convergent?

(a.) $\sum_{n=1}^{\infty}\frac{\sqrt{2n^2+3}}{5n^3+7}$
(b.) $\sum_{n=1}^{\infty}\frac{(n+1)^n}{n^{n+\frac{3}{2}}}$
(c.) $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{1}{n}\right)$
(d.) None of these

25. Consider the sequence of rational number $\{q_k\}_{k\geq 1}$ where $q_k = \sum_{n=1}^{k} \frac{1}{10^{n^2}}$ i.e., the sequence is $q_1 = .1$, $q_2 = .1001$, $q_3 = .100100001$ etc. Which of the following is true?

(a.) This sequence is bounded and convergent in $\mathbb{Q}$.

(b.) This sequence is not bounded.

(c.) This sequence is bounded, but not a Cauchy sequence.

[✓ (d.)] This sequence is bounded and Cauchy but not convergent in $\mathbb{Q}$

26. Let $\{a_0, a_1, a_2, \ldots\}$ be a sequence of real numbers. For any $k \geq 1$, let $s_n = \sum_{k=0}^{n} a_{2k}$. Which of the following statements are correct?

(a.) If $\lim_{n\to\infty} s_n$ exists, then $\sum_{m=0}^{\infty} a_m$ exists.

[✓ (b.)] If $\lim_{n\to\infty} s_n$ exists, then $\sum_{m=0}^{\infty} a_m$ need not exist.

(c.) If $\sum_{m=0}^{\infty} a_m$ exists, then $\lim_{n\to\infty} s_n$ exists.

[✓ (d.)] If $\sum_{m=0}^{\infty} a_m$ exists, then $\lim_{n\to\infty} s_n$ need not exist.

27. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then which of the following is NOT ture?

(a.) $\sum_{m=n}^{\infty} a_m \to 0$ as $n \to \infty$.

(b.) $\sum_{n=1}^{\infty} a_n \sin n$ is convergent.

(c.) $\sum_{n=1}^{\infty} e^{a_n}$ is divergent.

[✓ (d.)] $\sum_{n=1}^{\infty} a_n^2$ is divergent.

28. Using the fact that $\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $\sum_{1}^{\infty} \frac{1}{(2n+1)^2}$ equals

(a.) $\frac{\pi^2}{12}$

(b.) $\frac{\pi^2}{12} - 1$

(c.) $\frac{\pi^2}{8}$

(d.) $\frac{\pi^2}{8} - 1$

29. Pick out the series which are absolutely convergent:

[✓ (a.)] $\sum_{n=1}^{\infty} (-1)^n \frac{\cos n\alpha}{n^2}$ where $\alpha \in \mathbb{R}$ is a fixed real number.

[✓ (b.)] $\sum_{n=1}^{\infty} (-1)^n \frac{n \log n}{e^n}$

(c.) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

(d.) None of these

30. If $\sum a_n$ is series of positive & negative term and $\sum q_n, \sum p_n$ are series of negative & positive terms respectively and if $\sum a_n$ is conditionally convergent then

(a.) $\sum p_n$ is convergent but $\sum q_n$ not

(b.) $\sum p_n$ is divergent but $\sum q_n$ not

(c.) $\sum p_n$ & $\sum q_n$ both are convergent

[✓ (d.)] $\sum p_n$ & $\sum q_n$ both are divergent

31. Choose the correct answer, where $a_n \geq 0$

[✓ (a.)] If Σa_n is convergent then Σa_n^2 is also convergent

(b.) If Σa_n is convergent then $\Sigma \sqrt{a_n}$ is also convergent

[✓ (c.)] If Σa_n is convergent then $\Sigma \frac{a_n}{1+a_n}$ is also convergent

[✓ (d.)] If Σa_n is convergent then $\Sigma \frac{a_n}{1-a_n}$ (provided $a_n \neq 1$) is also convergent.

32. Let $\sum_{n=1}^{\infty} x_n$ be a series of real numbers. Which of the following is true?

(a.) If $\sum_{1}^{\infty} x_n$ is divergent, then $\langle x_n \rangle$ does not converge to 0

(b.) If $\sum_{n=1}^{\infty} x_n$ is convergent, then $\sum_{n=1}^{\infty} x_n$ is absolutely convergent

(c.) If $\sum_{n=1}^{\infty} x_n$ is convergent, then $x_n^2 \to 0$ as $n \to \infty$

(d.) If $x_n \to 0$, then $\sum_{n=1}^{\infty} x_n$ is convergent

33. If Σa_n be a convergent series and $a_n > 0$, $\forall n \in \mathbb{N}$ then the correct statement:

(a.) $\Sigma\sqrt{a_n\, a_{n+1}}$ is convergent

(b.) Σa_n^2 is convergent

(c.) $\Sigma \frac{\sqrt{a_n}}{n}$ is convergent

(d.) None of the above

34. Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be two sequences of real number such that $a_n = b_n - b_{n+1}$ for $n \in \mathbb{N}$ Then

(a.) Convergence of Σa_n implies the convergence of $\langle b_n \rangle$ not conversely

(b.) Convergence of Σa_n implies the convergence of Σb_n but not conversely

(c.) Convergence of Σa_n implies and implied by the convergence of $\langle b_n \rangle$

(d.) Convergence of Σb_n implies and implied by the convergence of $\langle a_n \rangle$

35. The alternating series $\sum_{n=1}(-1)^{n-1} u_n$ converges if $\{u_n\}$ is a/an

(a.) Decreasing sequence which converges to zero

(b.) Increasing sequence which converges to zero

(c.) Monotone sequence which converges to zero

(d.) Strictly monotone sequence which converges to zero

6. Which one of the following series is convergent?

(a.) $\frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \ldots..$ to ∞

(b.) $1\frac{1}{2} - 1\frac{1}{3} + 1\frac{1}{4} - 1\frac{1}{5} + \ldots..$ to ∞

(c.) $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots...$ to ∞

(d.) $x + x^2 + x^3 + x^4 + \ldots$ to a ∞, where $|x| < 1$

37. Select the incorrect statements

(a.) Finite set has infinite number of limit points

(b.) The sequence $1 + r + r^2 + \ldots + r^n$, where $-1 \le r < 1$ and $n \in \mathbb{N}$ is bounded above but not bounded below.

(c.) If $u_n > 0 \forall n$ then $\sum u_n$ & $\sum \frac{1}{u_n}$ converge or diverge together

(d.) Set of real number except the integers which are multiples of 2 is uncountable set

38. The set of all positive values of a for which the series $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \tan^{-1}\left(\frac{1}{n}\right)\right)^a$ converges, is

(a.) $\left(0,\ \frac{1}{3}\right]$

(b.) $\left(0,\ \frac{1}{3}\right)$

(c.) $\left[\frac{1}{3},\ \infty\right)$

(d.) $\left(\frac{1}{3},\ \infty\right)$

39. Let S be the series $\sum_{k=1}^{\infty}\frac{1}{(2k-1)2^{(2k-1)}}$ and T be the series $\sum_{k=3}^{\infty}\left(\frac{3k-4}{3k+2}\right)^{\frac{(k+1)}{3}}$ of real numbers. Then which one of the following is TRUE?

(a.) Both the series S and T are convergent

(b.) S is convergent and T is divergent

(c.) S is divergent and T is convergent

(d.) Both the series S and T are divergent

40. For $n \ge 1$, let

$$a_n = \begin{cases} n2^{-n}, & \text{if } n \text{ is odd} \\ -3^{-n}, & \text{if } n \text{ is even} \end{cases}$$

Which of the following statements is (are) TRUE?

(a.) The sequence $\{a_n\}$ converges

(b.) The sequence $\left\{|a_n|^{1/n}\right\}$ converges

(c.) The series $\sum_{n=1}^{\infty} a_n$ converges

(d.) The series $\sum_{n=1}^{\infty}|a_n|$ converges

ASSIGNMENT SHEET - 5

1. If $f(x)=1/(1-x), g(x)=f[f(x)]$ and $h(x)=f[g(x)]$, then what is $f(x)g(x)h(x)$ equal to ?

[✓ (a.) -1]

(b.) 0

(c.) 1

(d.) 2

2. What is the value of $\lim\limits_{n\to\infty}\dfrac{2^{n+1}+3^{n+1}}{2^n+3^n}$?

(a.) 1/3

(b.) 1

[✓ (c.) 3]

(d.) ∞

3. What is the value of $\lim\limits_{x\to 0}\dfrac{e^x-x-1}{x^2}$?

(a.) 0

[✓ (b.) 1/2]

(c.) 2

(d.) e

4. If D is the set of all real x such that $f(x)=1-e^{\left(\frac{1}{x}\right)-1}$ is positive, then what is D equal to?

[✓ (a.) $(-\infty,0)\cup(1,\infty)$]

(b.) $(-\infty,0)\cup[1,\infty)$

(c.) $(1,\infty)$

(d.) $(-\infty,1)$

5. If $\lim\limits_{x\to 0}\dfrac{(x+3\sin x-x^3-k\sinh x)}{1-\cos x+x^2-3x^3}$ exists then what is the value of k ?

(a.) -1

(b.) 2

(c.) 3

(d.) 4

6. What is the value of $\lim\limits_{x\to y}\dfrac{x^y-y^x}{x^x-y^y}$?

(a.) $\dfrac{1+\ln y}{1-\ln y}$

[✓ (b.) $\dfrac{1-\ln y}{1+\ln y}$]

(c.) $\dfrac{-1+\ln y}{1+\ln y}$

(d.) $\dfrac{-1-\ln y}{1-\ln y}$

7. **Assertion (A):** $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=0$

Reason (R): $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=\lim\limits_{x\to\infty}\sin x\lim\limits_{x\to\infty}\dfrac{1}{x}$

(a.) Both **A** and **R** are individually true and **R** is the correct explanation of **A**

(b.) Both **A** and **R** are individually true but R is not the correct explanation of **A**

[✓ (c.) **A** is true but **R** is false]

(d.) **A** is false but **R** is true

8. What is the value of $\lim\limits_{x\to\infty}\{\sin(1/x)+\cos(1/x)\}=?$

(a.) 0

(b.) $\dfrac{1}{e}$

(c.) e

[✓ (d.) 1]

9. Which one of the following functions is not well defined?

(a.) $\sqrt{(1+\sin x)}$

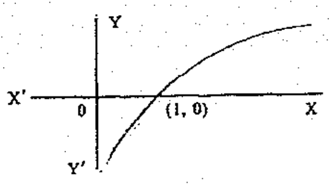
[✓ (b.) $\sqrt{(5\sec^2 x-4)}$]

(c.) $\sqrt{(3-\cos^2 x)}$

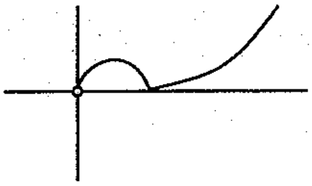
(d.) $\sqrt{\left(x^4+x^2+\dfrac{1}{100}\right)}$

10. Which one of the following graphs is the correct graph of the function $y(x) = x.\ln(x)$?

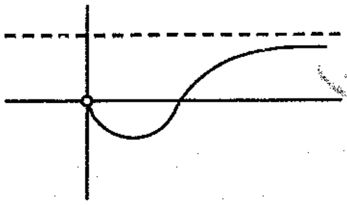
(a.)



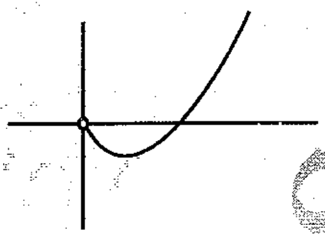
(b.)



[✓ (c.)]



(d.)



11. What are the value of a & b respectively, if $\lim\limits_{x\to 0}\frac{\sin ax - \ln\left(e^x \cos x\right)}{x \sin bx} = \frac{1}{2}$

(a.) 1, 1

(b.) 1, $\frac{1}{2}$

(c.) -1, 1

(d.) -1, $\frac{1}{2}$

12. **Assertion (A):** e^x cannot be expressed as sum of even and odd functions

Reason (R): e^x is neither even nor odd function

(a.) Both A and R are individually true and R is the correct explanation of A

(b.) Both A and R are individually true but R is not the correct explanation of A

(c.) A is true but R is false

(d.) A is false but R is true

13. Given the following limits:

1. $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$

2. $\lim\limits_{x\to \infty}\left(1+\frac{1}{x}\right) = 1$

3. $\lim\limits_{x\to 0}\frac{\log(x+1)}{x} = 1$

Which of these are correct?

(a.) 1, 2 and 3

[✓ (b.) 1 and 2]

(c.) 1 and 3

(d.) 2 and 3

14. If $f(x) = \frac{x-5}{x+5}, x \neq$ 5 then the domain of $f^{-1}(x)$, is

(a.) $\mathbb{R}$

[✓ (b.) $\mathbb{R} - \{1\}$]

(c.) $(-\infty, 1)$

(d.) $(1, \infty)$

15. $\lim\limits_{x\to\infty}\left(4^x + 5^x\right)^{\frac{1}{x}}$ equals to

(a.) 4

[✓ (b.) 5]

(c.) e

(d.) 5e

16. $\lim\limits_{x\to 0}\frac{e^x + e^{-x} - 2}{x^2}$ is equal to

[✓ (a.) 1]

(b.) – 1

(c.) 1/2

(d.) –1/2

17. Let $f: \mathbb{R} \to \mathbb{R}$ define $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x)\left(f(x) + f(-x)\right)$.

Then

(a.) g is even for all f

(b.) g is odd for all f

[✓ (c.) g is even if f is even]

(d.) g is even if f is odd

18. If α, β are the roots of the equation $ax^2+bx+c=0$, then $\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is equal to

(a.) 0

(b.) $\frac{a}{2}(\alpha-\beta)^2$

(c.) $-\frac{a}{2}(\alpha-\beta)^2$

[✓ (d.)] $\frac{a^2}{2}(\alpha-\beta)^2$

19. If $\lim_{x\to 0}\frac{x(1-\cos x)-ax^2\sin x}{x^5}$ exists and is finite, then the value of a must be

(a.) 1

(b.) $\frac{1}{2}$

(c.) $\frac{1}{3}$

(d.) $\frac{1}{4}$

20. Which of the following statement is true?

(a.) $\lim_{x\to\infty}\frac{\log x}{x^{1/2}}=0$ and $\lim_{x\to\infty}\frac{\log x}{x}=\infty$

(b.) $\lim_{x\to\infty}\frac{\log x}{x^{1/2}}=\infty$ and $\lim_{x\to\infty}\frac{\log x}{x}=0$

(c.) $\lim_{x\to\infty}\frac{\log x}{x^{1/2}}=0$ and $\lim_{x\to\infty}\frac{\log x}{x}=0$

(d.) $\lim_{x\to\infty}\frac{\log x}{x^{1/2}}=0$ but $\lim_{x\to\infty}\frac{\log x}{x}$ does not

21. Let $f:\mathbb{R}\to\mathbb{R}$ define $g:\mathbb{R}\to\mathbb{R}$ by $g(x)=f(x)\big(f(x)+f(-x)\big)$.

Then

(a.) g is even for all f

(b.) g is odd for all f

[✓ (c.)] g is even if f is even

(d.) g is even if f is odd

22. If $\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}=b$, where $\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}$ is finite ,then the values of a and b respectively will be

(a.) –2, -1

(b.) 2, 1

(c.) –2, 1

(d.) 2, -1

23. $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$ equals

[✓ (a.)] 1

(b.) 0

(c.) –1

(d.) -∞

24. Consider the following statements: if $f(x)=\begin{cases} x & 0\le x<1 \\ 3-x & 1\le x\le 2\end{cases}$ then

1. $\lim_{x\to 1^-} f(x)=1$

2. $\lim_{x\to 1^+} f(x)=2$

3. $\lim_{x\to 1} f(x)=2$

Of these statements

(a.) 1 and 3 are correct

(b.) 1, 2 and 3 are correct

[✓ (c.)] 1 and 2 are correct

(d.) 2 alone is correct

25. $\lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)}(a\ne 0, b\ne 0)$ is

[✓ (a.)] $\frac{a}{b}$

(b.) $\frac{b}{a}$

(c.) ab

(d.) does not exist

26. $\lim_{x\to\infty}\frac{ax+b}{cx}$ is

[✓ (a.)] a/c

(b.) ∞

(c.) B

(d.) none of the above

27. The correct value of $\lim_{x\to 0}\frac{x}{\sqrt{1-\cos x}}$

[✓ (a.) does not exist]

(b.) is $\sqrt{2}$

(c.) is $-\sqrt{2}$

(d.) is $1/\sqrt{2}$

28. The limit $\lim_{x\to 0^+}\frac{9}{x}\left(\frac{1}{\tan^{-1}x}-\frac{1}{x}\right)$

(a.) doesnot exist

(b.) 9

(c.) 0

(d.) 3

29. If X and Y are two non-empty finite sets and $f:X\to Y$ and $g:Y\to X$ are mappings such that $g\,o\,f:X\to X$ is a surjective (i.e., onto) map, then

[✓ (a.) f must be one-to-one]

(b.) f must be onto

(c.) g must be one-to-one

(d.) X and Y must have the same number of elements.

30. Let $P(x)$ be a non-constant polynomial such that $P(n)=P(-n)$ for all $n\in\mathbb{N}$. Then $P'(0)$

(a.) Equals 1

[✓ (b.) Equals 0]

(c.) Equals -1

(d.) Can not be determined from the given data.

31. If $\lim_{x\to 0}\left(\frac{1+cx}{1-cx}\right)^{\frac{1}{x}}=4$, then $\lim_{x\to 0}\left(\frac{1+2cx}{1-2cx}\right)^{\frac{1}{x}}$ is

(a.) 2

(b.) 4

[✓ (c.) 16]

(d.) 64

32. Let $f(x)=e^{\left(e^{x}\right)}$ and define $g(x)=f(x+1)-f(x)$. Then, as $x\to\infty$, the function $g(x)$ converges to

(a.) 0

(b.) 1

(c.) e

(d.) e^{e}

33. Consider the functions $f,g:\mathbf{Z}\to\mathbf{Z}$ defined by $f(n)=3n+2$ and $g(n)=n^2-5$

(a.) Neither f and g is a one-to-one function

(b.) The function f is one-to-one, but not g

(c.) The function g is one-to-one, but not f

(d.) Both f and g are one-to-one functions

34. Let T be the graph of the function

$$f(x)=\begin{cases}1+x, & -1\le x\le 0;\\ 1-x, & 0\le x\le 1\end{cases}$$

Then the reflection of T in the line $y=0$ is given by the graph of $g(x)$ where

[✓ (a.) $g(x)=\begin{cases}-1-x, & -1\le x\le 0\\ -1+x, & 0\le x\le 1\end{cases}$]

(b.) $g(x)=\begin{cases}-1+x, & -1\le x\le 0\\ -1-x, & 0\le x\le 1\end{cases}$

(c.) $g(x)=\begin{cases}-1-x, & -1\le x\le 0\\ 1-x, & 0\le x\le 1\end{cases}$

(d.) $g(x)=\begin{cases}1-x, & -1\le x\le 0\\ -1-x, & 0\le x\le 1\end{cases}$

35. Let a,b,c,d be rational number with $ad-bc\neq 0$. then the function $f:R\setminus Q\to R$ defined by $f(x)=\frac{ax+b}{cx+d}$ is

(a.) onto but not one-one.

[✓ (b.) one-one but not onto.]

(c.) neither one-one nor onto.

(d.) both one-one and onto.

36. Let $f:(0,\infty)\to R$ be the function defined by $f(x):=\frac{e^{x}}{x^{x}}$. Then the $\lim_{x\to\infty}f(x)$

(a.) does not exist.

[✓ (b.) exists and is 0]

(c.) exists and is 1

(d.) exists and is e

37. $\lim\limits_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n$ equals

(a.) 1

(b.) $e^{-1/2}$

(c.) e^{-2}

(d.) e^{-1}

38. $\lim\limits_{n\to\infty}\frac{1}{n^4}\sum\limits_{j=0}^{2n-1} j^3$ equals

[✓ (a.) 4]

~~(b.) 16~~

(c.) 1

(d.) 8

39. Which of the following are true for function $f(x)=\sin(x)\sin\left(\frac{1}{x}\right), x\in(0,1)$?

[✓ (a.) $\underline{\lim}_{x\to 0} f(x)=\overline{\lim}_{x\to 0} f(x)$]

(b.) $\underline{\lim}_{x\to 0} f(x)<\overline{\lim}_{x\to 0} f(x)$

(c.) $\underline{\lim}_{x\to 0} f(x)=1$

[✓ (d.) $\overline{\lim}_{x\to 0} f(x)=0$]

40. The number of distinct real roots of the equation $x^9+x^7+x^5+x^3+x+1=0$ is

[✓ (a.) 1]

(b.) 3

(c.) 5

(d.) 9

ASSIGNMENT SHEET - 6

1. Let $f:X\to X$ such that $f(f(x))=x$ for all $x\in X$. Then
   (a.) f is one-to-one and onto.
   (b.) f is one-to-one but not onto.
   (c.) f is onto but not one-to-one.
   (d.) f need not be either one-to-one or onto.

2. A polynomial of odd degree with real coefficients must have
   (a.) at least one real root.
   (b.) no real root.
   (c.) only real roots.
   (d.) at least one root which is not real.

3. Consider the following sets of functions on $\mathbb{R}$, $W=$ The set of constant functions on $\mathbb{R}$, $X=$ The set of polynomial functions on $\mathbb{R}$, $Y=$ The set of continuous functions on $\mathbb{R}$, $Z=$ The set of all functions on $\mathbb{R}$ Which of these sets has the same cardinality as that of $\mathbb{R}$?
   (a.) Only W
   (b.) Only W and X
   (c.) Only W, X and Z
   (d.) Only W, X, Y

4. If $f:[0,1]\to(0,1)$ is a continuous mapping then which of the following is NOT true?
   (a.) $F\subseteq[0,1]$ is a closed set implies $f(F)$ is closed in $\mathbb{R}$.
   (b.) If $f(0)<f(1)$ then $f([0,1])$ must be equal to $[f(0),f(1)]$.
   (c.) There must exist $x\in(0,1)$ such that $f(x)=x$.
   (d.) $f:([0,1])\neq(0,1)$.

5. In which of the following cases, there is no continuous function f from the set S **onto** the set T?
   (a.) $S=[0,1], T=\mathbb{R}$
   (b.) $S=(0,1), T=\mathbb{R}$
   (c.) $S=(0,1), T=(0,1]$
   (d.) $S=\mathbb{R}, T=(0,1)$

6. Let $A_n\subseteq\mathbb{R}$ for $n\geq 1$, and $\chi_n:\mathbb{R}\to\{0,1\}$ be the function $\chi_n(x)=\begin{cases}0 & \text{if } x\notin A_n\\ 1 & \text{if } x\in A_n\end{cases}$,let $g(x)=\limsup_{n\to\infty}\chi_n(x)$ and $h(x)=\liminf_{n\to\infty}\chi_n(x)$
   (a.) If $g(x)=h(x)=1$, then there exists m such that for all $n\geq m$ we have $x\in A_n$
   (b.) If $g(x)=1$ and $h(x)=0$, then there exist m such that for all $n\geq m$ we have $x\in A_n$
   (c.) If $g(x)=1$ and $h(x)=0$ then there exists a sequence $n_1,n_2,\ldots$ of distinct integers such that $x\in A_{n_k}$ for all $k\geq 1$
   (d.) If $g(x)=h(x)=0$ then there exists m such that for all $n\geq m$ we have $x\notin A_n$.

7. Let f be a monotonicly non-decreasing real-valued function on $\mathbb{R}$. Then
   (a.) $\lim_{x\to a}f(x)$ exists at each point a
   (b.) If $a<b$, then $\lim_{x\to a^+}f(x)\leq\lim_{x\to b^-}f(x)$
   (c.) f is an unbounded function
   (d.) The function $g(x)=e^{-f(x)}$ is a bounded function

8. Let $f:\mathbb{R}\to\mathbb{R}$ be a strictly increasing continuous function. If $\{a_n\}$ is sequence in $[0,1]$ then the sequence $\{f(a_n)\}$ is
   (a.) Increasing
   (b.) Bounded
   (c.) Convergent
   (d.) Not necessarily bounded

9. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=[x^2]$. The points of discontinuity of f are
   (a.) Only the integral points
   (b.) All rational numbers
   (c.) $\{\pm\sqrt{n}: n$ is a non-negative integer$\}$
   (d.) All real numbers

10. A monotonic function

(a.) Is always continuous

(b.) Is continuous only if it intermediate value property

(c.) Can be nowhere continuous

(d.) Can be discontinuous at finitely many points

11. Which one of the following functions has exactly two points of discontinuity?

(a.) $f(x)=\begin{cases}1 & \text{if } x>0\\ 0 & \text{if } x\le 0\end{cases}$

(b.) $f(x)=\begin{cases}1 & \text{if } 0\le x\le 1\\ 0 & \text{otherwise}\end{cases}$

(c.) $f(x)=\begin{cases}x & \text{if } 0\le x\le 1\\ 0 & \text{otherwise}\end{cases}$

(d.) $f(x)=\begin{cases}1 & \text{if } x \text{ is rational}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$

12. Let $f:[a,\ b]\to R$ be a continuous function and let $f(a)<f(b)$. Then by intermediate value theorem

(a.) $f([a,\ b])=[f(a),f(b)]$

(b.) $f([a,b])\supseteq[f(a),f(b)]$

(c.) $f([a,b])\subseteq[f(a),f(b)]$

(d.) $f([a,b])\neq[f(a),f(b)]$

13. Let $f:\mathbf{R}\to\mathbf{R}$ be defined by $f(t)=t^2$ and let U be any non-empty open subset of $\mathbf{R}$. Then

(a.) $f(U)$ is open

(b.) $f^{-1}(U)$ is open

(c.) $f(U)$ is closed

(d.) $f^{-1}(U)$ is closed

14. Let $f:[0,1]\to\mathbb{R}$ be the continuous function defined by $f(x)=\dfrac{(x-1)(x-2)}{(x-3)(x-4)}$. Then the maximal subset of $\mathbb{R}$ on which f has a continuous extension is

(a.) $(-\infty,3)$

(b.) $(-\infty,3)\cup(4,\infty)$

(c.) $\mathbb{R}\setminus\{3,4\}$

(d.) $\mathbb{R}$

15. The continuous function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=(x^2+1)^{2003}$ is

(a.) Onto but not one-one

(b.) One-one but not onto

(c.) Both one-one and onto

(d.) Neither one-one nor onto

16. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. If $f(\mathbb{Q})\subseteq\mathbb{N}$, then

(a.) $f(\mathbb{R})=\mathbb{N}$

(b.) $f(\mathbb{R})\subseteq\mathbb{N}$ but f need not be constant

(c.) f is unbounded

(d.) f is a constant function

17. Let $f:[a,b]\to\mathbb{R}$ be a monotonic function. Then

(a.) f is continuous

(b.) f is discontinuous at at most two points

(c.) f is discontinuous at at finitely many points

(d.) f is discontinuous at at most countable points

18. Let $f:[0,10)\to[0,10]$ be a continuous mapping. Then

(a.) f need not have any fixed point

(b.) f has at least 10 fixed points

(c.) f has at least 9 fixed points

(d.) f has at least one fixed point

19. Let $S=[0,1]\cup[2,3)$ and let $f:S\to\mathbb{R}$ be defined by $f(x)=\begin{cases}2x & \text{if } x\in[0,1],\\ 8-2x & \text{if } x\in[2,3).\end{cases}$ If $T=\{f(x):x\in S\}$, then the inverse function $f^{-1}:T\to S$

(a.) does NOT exist

(b.) exists and is continuous

(c.) exists and is NOT continuous

(d.) exists and is monotonic

20. Let f be a continuous function from $[0,4]$ to $[3,9]$. Then

(a.) There must be an x such that $f(x)=4$

(b.) There must be an x such that $3f(x)=2x+6$

(c.) There must be an x such that $2f(x)=3x+6$

(d.) There must be an x such that $f(x)=x$

21. Let f_1 and f_2 be two real valued functions defined on the real line. Define two functions g and h by $g(x)=\max\{f_1(x),f_2(x)\}$ and $h(x)=\min\{f_1(x),f_2(x)\}$ .Then $g(x^2)+h(x)^2+3g(x)h(x)$ $=f_1(x)^2+f_2(x)^2+3f_1(x)f_2(x)$ holds for all $x\in\mathbb{R}$

(a.) Always

(b.) Only if $f_1(x)=f_2(x)$ for all x in $\mathbb{R}$

(c.) Only f_1 and f_2 are both positive functions or both negative functions

(d.) Only if at least one of the functions f_1 and f_2 is identically zero.

22. Let f be a strictly monotonic continuous real valued function defined on $[a,b]$ such that $f(a)<a$ and $f(b)>b$. Then which one of the following is TRUE?

(a.) There exists exactly one $c\in(a,b)$ such that $f(c)=c$

(b.) There exist exactly two points $c_1,c_2\in(a,b)$ such that $f(c_i)=c_i, i=1,2$

(c.) There exists no $c\in(a,b)$ such that $f(c)=c$

(d.) There exist infinitely many points $c\in(a,b)$ such that $f(c)=c$

23. Let X and Y be two non-empty sets and let $f:X\to Y$, $g:Y\to X$ be two mappings. If both f and g are injective (i.e., one-to-one) then

(a.) X and Y must be infinite sets.

(b.) $g=f^{-1}$ always.

(c.) One of $f\circ g:Y\to X$ and $g\circ f:X\to Y$ is always bijective (one-to-one and onto).

(d.) There exists a bijective mapping $h:X\to Y$

24. Define a function f on the real line by

$$f(x)=\begin{cases} x-[x]-\frac{1}{2} & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer} \end{cases}$$

Then which of the following is true:

(a.) f is periodic with period 1, i.e., $f(x+1)=f(x)$ for all x

(b.) f is continuous

(c.) f is one-to-one

(d.) $\lim_{x\to a} f(x)$ exists for all $a\in\mathbb{R}$.

25. Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f(x+y)=f(x)f(y)$ for all $x,y\in\mathbb{R}$ and $f(x)=1+xg(x)$ where $\lim_{x\to 0} g(x)=1$. Then the function $f(x)$ is

(a.) e^x

(b.) 2^x

(c.) A non-constant polynomial

(d.) Equal to 1 for all $x\in\mathbb{R}$.

26. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying $f\circ f=f$. Then

(a.) f must be constant

(b.) $f(x)=x$ for all x in the range of f

(c.) f must be a non-constant polynomial

(d.) There is no such function

27. Let $f:[0,1]\to[0,1]$ be continuous and $f(0)=0, f(1)=1$. Then, f is necessarily

(a.) injective, but not surjective.

(b.) surjective, but not injective.

(c.) bijective.

(d.) surjective.

28. Let p be a polynomial of degree $2n+1$ with real coefficients. We say that a real number a is a fixed point of p if $p(a)=a$. Then, p has

(a.) exactly $2n+1$ fixed points

(b.) at least one fixed point

(c.) at most one fixed point

(d.) n fixed points

29. Define the function $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=\begin{cases} ax^2+b, & if\ x\le 1 \\ cx+1, & if\ x>1 \end{cases}$ We want to find appropriate values of a, b and c such that

1. f is increasing in the interval $(0,\infty)$;
2. f' is continuous on $\mathbf{R}$.

Which of the following is the correct statement about the values of (a,b,c) for which both conditions (1) and (2) are satisfied?

(a.) (3, 2, 6) is the only possible value

(b.) There are finitely many values of (a,b,c)

(c.) $(-2,-3,-4)$ is one of the values

(d.) There are infinitely many values of (a,b,c)

30. The limit $\lim_{x\to 0}\frac{3^x-2^x}{4^x-3^x}$ equals

(a.) $\frac{3}{4}$

(b.) $\log_{4/3}\left(\frac{3}{2}\right)$

(c.) $\log_e\left(\frac{3}{2}\right)$

(d.) $\frac{2}{3}$

31. The equation $x^2=x\sin x+\cos x$ is true for

(a.) no real value of x

(b.) exactly one real value of x

(c.) exactly two real values of x

(d.) infinitely many real values of x

32. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous with $f(0)=f(1)=0$. Which of the following is not possible?

(a.) $f([0,\ 1])=\{0\}$

(b.) $f([0,\ 1])=[0,\ 1)$

(c.) $f([0,\ 1])=[0,\ 1]$

(d.) $f([0,\ 1])=\left[-\frac{1}{2},\ \frac{1}{2}\right]$

33. Consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\begin{cases} 2x, & \text{if } x \text{ is rational} \\ 3-x, & \text{if } x \text{ is irrational} \end{cases}$ Then f is continuous

(a.) only at x = 1.

(b.) only at x = 2.

(c.) only at x = 0.

(d.) at no point of **R**.

34. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=|x+1|-|x|$, then the range of f is

(a.) The whole of $\mathbb{R}$.

(b.) The closed interval $[-1,1]$

(c.) The unbounded subset of $\mathbb{R}$.

(d.) None of the above.

35. How many zeroes are there for the function $x^2-5|x|+6$?

(a.) 3

(b.) 4

(c.) 6

(d.) None of these

36. Let $f(x)=\cos^{-1}x$. Then f is one-one and onto if the domain and range are specified respectively as

(a.) [-1,1] and $(-\infty,\infty)$

(b.) [-1,1] and $(0,2\pi)$

(c.) [-1,1] and $(0,\pi/2)$

(d.) [-1,1] and $[0,\pi]$

37. A bijection is a map that is both one-one and onto. It is given that $f:\mathbb{R}\to\mathbb{R}$ is not a bijection. Which of the following must be true?

(a.) If *f* is not one-one, then *f* is not onto.

(b.) If *f* is not onto, then *f* is not one-one.

(c.) *f* is neither one-one nor onto.

(d.) If *f* is one-one, then *f* is not onto.

38. Define $f:\mathbb{R}\to\mathbb{R}$ as follows: $f(x)=\begin{cases} 1, & \text{if } x \text{ is rational} \\ \frac{\sin x}{x}, & \text{if } x \text{ is irrational} \end{cases}$ Then

(a.) *f* is continuous everywhere

(b.) *f* is continuous only at $x=0$.

(c.) *f* is continuous all rational points.

(d.) *f* is continuous at all irrational points.

39. Let $f,g:\mathbb{R}\to\mathbb{R}$ be continuous functions whose graphs do not intersect. Then for which function below the graph lies entirely on one side of the X-axis (a.) f

(b.) $g+f$

(c.) $g-f$

(d.) $g.f$

40. $f(x)=e^{x}-e^{-x}, g(x)=e^{x}+e^{-x}$. Then

(a.) Both f and g are even functions

(b.) Both f and g are odd functions

(c.) f is odd, g is even

(d.) f is even, g is odd

41. Let $F:\mathbb{R}\to\mathbb{R}$ be a monotone function. Then

(a.) F has no discontinuities.

(b.) F has only finitely many discontinuities.

(c.) F can have at most countably many discontinuities.

(d.) F can have uncountably many discontinuities

42. Let X be a set and $f,g:X\to X$ be functions. We can say that fog is bijective if

(a.) at least one of f, g is bijective

(b.) both f and g are bijective

(c.) f is 1–1 and g is onto

(d.) f is onto and g is 1-1

43. Let $f,g:\mathbb{R}\to\mathbb{R}$ be functions. We can conclude that $h(x)\le f(x)\forall x\in\mathbb{R}$ if we define $h:\mathbb{R}\to\mathbb{R}$ as

(a.) $\min\{g(x), f(x)+g(x)\}$

(b.) $\min\{f(x), f(x)+g(x)\}$

(c.) $\max\{g(x), f(x)+g(x)\}$

(d.) $\max\{f(x), f(x)+g(x)\}$

44. Let X be a non-empty set, $f:X\to X$ be a function and let $A,B\subset X$. Then the identity $f(A\cap B)=f(A)\cap f(B)$ is

(a.) Always holds

(b.) holds if f is 1-1

(c.) holds if f is onto

(d.) holds if $A\cup B=X$

45. The range of the function $f(x)=\dfrac{x}{\sqrt{x^{2}+1}}, x\in\mathbb{R}$ is

(a.) [-1, 1]

(b.) [-1, 1)

(c.) (-1, 1]

(d.) None of these

ASSIGNMENT SHEET - 7

1. The value of $\lim_{x\to 0}\frac{1}{x}\int_x^{2x} e^{-t^2}\,dt$

(a.) does not exist.

(b.) is infinite.

(c.) exists and equals 1.

(d.) exists and equals 0.

2. Let f be a bounded function on $\mathbb{R}$ and $a\in\mathbb{R}$. For $\delta>0$, let $\omega(a,\delta)=\sup|f(x)-f(a)|,\quad x\in[a-\delta,a+\delta]$. Then

(a.) $\omega(a,\delta_1)\le\omega(a,\delta_2)$ if $\delta_1\le\delta_2$.

(b.) $\lim_{\delta\to0+}\omega(a,\delta)=0$ for all $a\in\mathbb{R}$.

(c.) $\lim_{\delta\to0+}\omega(a,\delta)$ need not exist.

(d.) $\lim_{\delta\to0+}\omega(a,\delta)=0$ if and only if f is continuous at a.

3. Let f be a continuously differentiable function on $\mathbb{R}$ suppose that $L=\lim_{X\to\infty}(f(x)+f'(x))$ exists. If $0<L<\infty$ then which of the following statements is/are correct?

(a.) If $\lim_{x\to\infty}f'(x)$ exists, then it is 0

(b.) If $\lim_{x\to\infty}f(x)$ exists it is L

(c.) If $\lim_{x\to\infty}f'(x)$ exists then $\lim_{x\to\infty}f(x)=0$

(d.) If $\lim_{x\to\infty}f(x)$ exists then $\lim_{x\to\infty}f'(x)=L$

4. Let $A\subseteq R$ and $f:A\to R$ be given by $f(x)=x^2$. Then f is uniformly continuous if

(a.) A is a bounded subset of R

(b.) A is a dense subset of R

(c.) A is an unbounde~~d and~~ connected subset of R

(d.) A is an unbounded and open subset of R

5. Let $P(x)=\left(\frac{5}{13}\right)^x+\left(\frac{12}{13}\right)^x-1$ for all $x\in\mathbb{R}$. Then which of the following statement(s) is(are) TRUE?

(a.) The equation $P(x)=0$ has exactly one solution in $\mathbb{R}$

(b.) $P(x)$ is strictly increasing for all $x\in\mathbb{R}$

(c.) The equation $P(x)=0$ has exactly two solution $\mathbb{R}$

(d.) $P(x)$ is strictly decreasing for all $x\in\mathbb{R}$

6. The function $f:\mathbb{R}\to\mathbb{R}$ is given by $f(x)=e^{|x|+x^2}+|x^2-1|$ Which of the following is true about the function f?

(a.) It is not differentiable exactly at three points of $\mathbb{R}$.

(b.) It is not differentiable at $x=0$.

(c.) It is differentiable at $x=2$.

(d.) It is not differentiable at $x=1$ and $x=-1$.

7. The function $f(x)=a_0+a_1|x|+a_2|x|^2+a_3|x|^3$ is differentiable at $x=0$

(a.) For no values of a_0,a_1,a_2,a_3

(b.) For any value of a_0,a_1,a_2,a_3

(c.) Only if $a_1=0$

(d.) Only if both $a_1=0$ and $a_3=0$.

8. Suppose $f:\mathbb{R}\to\mathbb{R}$ is a function that satisfies $|f(x)-f(y)|\le|x-y|^\beta,\ \beta>0$. Which of the following is correct?

(a.) If $\beta=1$ then f is differentiable

(b.) If $\beta>0$ then f is uniformly continuous

(c.) If $\beta>1$ then f is a constant function

(d.) f must be a polynomial

9. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function with period $\rho>0$. Then $g(x)=\int_x^{x+p} f(t)\,dt$ is a

(a.) Constant function

(b.) Continuous function

(c.) Continuous function but not differentiable

(d.) Neither continuous nor differentiable

10. Let A be the set of rational numbers in the open interval $(0,\ 7)$ and $f:A\to\mathbb{R}$ be a uniformly continuous function. Which of the following are true?

(a.) f is bounded

(b.) f is necessarily a constant function

(c.) f is differentiable on $(0,\ 7)$

(d.) f is differentiable at all the rational points in $(0,\ 7)$

11. Let f be a twice differentiable function on $\mathbb{R}$. Given that $f''(x)>0$ for all $x\in\mathbb{R}$,

(a.) $f(x)=0$ has exactly two solution on $\mathbb{R}$

(b.) $f(x)=0$ has a positive solution if $f(0)=0$ and $f'(0)=0$

(c.) $f(x)=0$ has no positive solution if $f(0)=0$ and $f'(0)>0$

(d.) $f(x)=0$ has no positive solution if $f(0)=0$ and $f'(0)<0$

12. Suppose $f:\mathbb{R}\to\mathbb{R}$ is a differentiable function. Then which of the following statements are necessarily true?

(a.) If $f'(x)\le r<1$ for all $x\in\mathbb{R}$, then f has at least one fixed point.

(b.) If f has a unique fixed point, then $f'(x)\le r<1$ for all $x\in\mathbb{R}$

(c.) If f has a unique fixed point, then $f'(x)\ge r>-1$ for all $x\in\mathbb{R}$

(d.) None of these

13. Let $f(x)=\sin x-x+\frac{x^3}{3!}$ and $g(x)=\cos x-1+\frac{x^2}{2!}$ for $x\in\mathbb{R}$. Which of the following statements are correct?

(a.) $f(x)\ge 0$ for all $x>0$

(b.) g is an increasing function on $[0,\ \infty)$

(c.) g is an decreasing function on $[0,\ \infty)$

(d.) f is a decreasing function on $[0,\ \infty)$

14. Let $I=[0,1]\subset R$. For $x\in R$, let $\varphi(x)=\operatorname{dist}(x,I)=\inf\{|x-y|:y\in I\}$. Then

(a.) $\varphi(x)$ is discontinuous somewhere on R

(b.) $\varphi(x)$ is continuous on R but not continuously differentiable exactly at $x=0$

(c.) $\varphi(x)$ is continuous on R but not continuously differentiable exactly at $x=0$ and $x=1$

(d.) $\varphi(x)$ is differentiable on R.

15. Consider the function $f(x)=\cos(|x-5|)+\sin(|x-3|)+|x+10|^3-(|x|+4)^2$ At which 0f the following point is f not differentiable?

(a.) $x=5$

(b.) $x=3$

(c.) $x=-10$

(d.) $x=0$

16. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=\begin{cases} x^2 & \text{if } x<0, \\ 2x+x^2 & \text{if } x\ge 0 \end{cases}$ Then which of the following statements are correct?

(a.) $f''(x)=2$ for all $x\in\mathbb{R}$

(b.) $f''(0)$ does not exist

(c.) $f'(x)$ exists for each $x\neq 0$

(d.) $f'(0)$ does not exist

17. Consider the function $f(x)=|\cos x|+|\sin(2-x)|$. At which of the following points is f not differentiable?

(a.) $\left\{(2n+1)\frac{\pi}{2}:n\in\mathbf{Z}\right\}$

(b.) $\{n\pi:n\in\mathbf{Z}\}$

(c.) $\{n\pi+2:n\in\mathbf{Z}\}$

(d.) $\left\{\frac{n\pi}{2}:n\in\mathbf{Z}\right\}$

18. Let
$F=\{f:\mathbf{R}\to\mathbf{R}:|f(x)-f(y)|\le K|(x-y)|^{\alpha}\}$
For all $x,y\in\mathbf{R}$ and for some $\alpha>0$ and some $K>0$. Which of the following is/are true?

(a.) Every $f\in F$ is continuous

(b.) Every $f\in F$ is uniformly continuous

(c.) Every differentiable function f is in F

(d.) Every $f\in F$ is differentiable

19. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by
$f(x)=x^2\sin\frac{1}{x^2},x\neq 0$, $f(0)=0$

(a.) f is differentiable everywhere with continuous derivative

(b.) f is differentiable everywhere other than the point 1

(c.) f is nowhere differentiable

(d.) f is differentiable but its derivative is not continuous

20. Let $f:(0,1)\to\mathbb{R}$ be continuous. Suppose that $|f(x)-f(y)|\le|\cos x-\cos y|$ for all $x,y\in(0,1)$. Then

(a.) f is discontinuous at least at one point in $(0,1)$.

(b.) f is continuous everywhere on $(0,1)$ but not uniformly continuous on $(0,1)$.

(c.) f is uniformly continuous on $(0,1)$.

(d.) $\lim_{x\to 0}f(x)$ exists.

21. The function $f(x)=|x|+3$ is

(a.) Continuous as well as differentiable on $\mathbb{R}$

(b.) Continuous on $\mathbb{R}$ but not differentiable anywhere on $\mathbb{R}$

(c.) Continuous on $\mathbb{R}$ and differentiable at all but finitely many points on $\mathbb{R}$

(d.) Non continuous on $\mathbb{R}$

22. Let $f(x)=|\sin\pi x|$, $x\in\mathbb{R}$ .Then

(a.) f is continuous nowhere

(b.) f is continuous everywhere and differentiable nowhere

(c.) f is continuous everywhere and differentiable everywhere except at integral value of x

(d.) f is differentiable everywhere

23. The function $f(x)=1-|1-x|$ on $\mathbb{R}$ is

(a.) Not continuous

(b.) Continuous but not differentiable

(c.) Differentiable but not continuous

(d.) Differentiable at only one point

24. $f:[0,1]\to\mathbb{R}$ is a function. Which of the following is possible?

(a.) f is differentiable but not continuous

(b.) f is unbounded and continuous

(c.) f is differentiable but its derivative is not continuous

(d.) f is continuous but not integrable

25. Let $f=\mathbb{R}\to\mathbb{R}$ be differentiable. Which of these will follow from mean value theorem?

(a.) For all a,b in $\mathbb{R}$, if $a<c<b$ then $\frac{f(b)-f(a)}{b-a}=f'(c)$

(b.) For some a,b in $\mathbb{R}$ and for all c in (a,b), $\frac{f(b)-f(a)}{b-a}=f'(c)$

(c.) For all a,b in $\mathbb{R}$, there is some c in (a,b) such that $\frac{f(b)-f(a)}{b-a}=f'(c)$

(d.) For all a,b in $\mathbb{R}$, there is a unique element c in (a,b) then the $\frac{f(b)-f(a)}{b-a}=f'(c)$

26. The function $f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right) & x\neq 0 \\ 0 & x=0 \end{cases}$ is

(a.) Differentiable at 0 but not continuous

(b.) Has 2nd derivative at the origin

(c.) Continuous at the origin but not differentiable

(d.) Neither continuous nor differentiable at the origin

27. The set of points where $f(x)=|\sin x|$ is not differentiable is

(a.) Empty

(b.) $\{0\}$

(c.) $\{k\pi : k\in \mathbb{Z}\}$

(d.) $\left\{\frac{k\pi}{2} : k\in \mathbb{Z}\right\}$

28. Let $f:(0,2)\to R$ be defined by $f(x)=\begin{cases} x^2 & \text{If x is rational} \\ 2x-1 & \text{If x is irrational} \end{cases}$ Then

(a.) f is differentiable exactly at one point

(b.) f is differentiable exactly at two points

(c.) f is not differentiable at any point in (0,2)

(d.) f is differentiable at every point in (0,2)

29. $f:\mathbb{R}\to\mathbb{R}$ is such that $f(0)=0$ and $\left|\frac{df}{dx}(x)\right|\leq 5$ for all x. We can conclude that $f(1)$ is in

(a.) $(5,6)$

(b.) $[-5,5]$

(c.) $(-\infty,-5)\cup(5,\infty)$

(d.) $[-4,4]$

30. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function such that $\sup_{x\in\mathbb{R}}|f'(x)|<\infty$. Then

(a.) f maps a bounded sequence to a bounded sequence.

(b.) f maps a Cauchy sequence to a Cauchy sequence.

(c.) f maps a convergent sequence to a convergent sequence.

(d.) f is uniformly continuous.

31. Let $f:\mathbb{R}\to\mathbb{R}$ be a twice continuously differentiable function, with $f(0)=f(1)=f'(0)=0$. Then

(a.) f'' is the zero function.

(b.) $f''(0)$ is zero.

(c.) $f''(x)=0$ for some $x\in(0,1)$.

(d.) f'' never vanishes.

32. Let $f:[0,\infty)\to[0,\infty)$ be a continuous function. Which of the following is correct?

(a.) There is $x_0\in[0,\infty)$ such that $f(x_0)=x_0$

(b.) If $f(x)\leq M$ for all $x\in[0,\infty)$ for some $M>0$, then there exists $x_0\in[0,\infty)$ such that $f(x_0)=x_0$.

(c.) If f has a fixed point, then it must be unique.

(d.) f does not have a fixed point unless it is differentiable on $(0,\infty)$.

33. Take the closed interval $[0,1]$ and open interval (1/3,2/3). Let $k=[0,1]\setminus(1/3,2/3)$. For $x\in[0,1]$ define $f(x)=d(x,K)$ where $f(x)=d(x,K)=\inf\{|x-y| \mid y\in K\}$. Then

(a.) $f:[0,1]K\to\mathbb{R}$ is differentiable at all points of (0,1).

(b.) $f:[0,1]\to\mathbb{R}$ is not differentiable at 1/3 and 2/3.

(c.) $f:[0,1]\to\mathbb{R}$ is not differentiable at 1/2.

(d.) $f:[0,1]\to\mathbb{R}$ is not continuous

34. Let $X=(0,1)\cup(2,3)$ be an open set in R. Let f be a continuous function on X such that the derivative $f'(x)=0$ for all x. Then the range of f has

(a.) Uncountable number of points
(b.) Count ably infinite number of points
(c.) At most 2 points
(d.) At most 1 point

35. If $f(x)$ is real valued function defined on $[0,\infty]$ such that $f(0)=0$ and $f''(x)>0$ for all x, then the function $h(x)=\frac{f(x)}{x}$ is

(a.) Increasing in $[0,\infty]$
(b.) Decreasing in $[0,1]$
(c.) Increasing in $[0,1]$ and decreasing in $[1,\infty]$
(d.) Decreasing in $[0,1]$ and increasing in $[1,\infty]$

36. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function and $f(x+1)=f(x)$ for all $x\in\mathbb{R}$. Then

(a.) f is bounded above, but not bounded below.
(b.) f is bounded above and below, but may not attain its bounds.
(c.) f is bounded above and below and f attains its bounds.
(d.) f is uniformly continuous

37. Decide which of the following functions are uniformly continuous on $(0,1)$.

(a.) $f(x)=e^{x}$
(b.) $f(x)=x$
(c.) $f(x)=\tan\left(\frac{\pi x}{2}\right)$
(d.) $f(x)=\sin(x)$

38. Which of the following statements is (are) on the interval $\left(0,\frac{\pi}{2}\right)$?

(a.) $\cos x<\cos(\sin x)$
(b.) $\tan x<x$
(c.) $\sqrt{1+x}<1+\frac{x}{2}-\frac{x^2}{8}$
(d.) $\frac{1-x^2}{2}<\ln(2+x)$

39. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function with $f(0)=0$ If for all $x\in\mathbb{R}, 1<f^{1}(x)<2$, then Which one of the following statements is true on $(0,\infty)$?

(a.) f is unbounded
(b.) f is increasing and bounded
(c.) f has at least one zero
(d.) f is periodic

40. Let $f(x)=\frac{1}{1+|x|}+\frac{1}{1+|x-1|}$ for all $x\in[-1,1]$. Then which one of the following is TRUE?

(a.) Maximum value of $f(x)$ is $\frac{3}{2}$
(b.) Minimum value of $f(x)$ is $\frac{1}{3}$
(c.) Maximum of $f(x)$ occurs at $x=\frac{1}{2}$
(d.) Minimum of $f(x)$ occurs at $x=1$

ASSIGNMENT SHEET - 8

1. Let a be a positive real number. Which of the following integrals are convergent?

(a.) $\int_0^a \frac{1}{x^4}dx$.

(b.) $\int_0^a \frac{1}{\sqrt{x}}dx$.

(c.) $\int_4^\infty \frac{1}{x\log_e x}dx$.

(d.) $\int_5^\infty \frac{1}{x(\log x)^2}dx$.

2. Let k be a positive integer. The radius of convergence of the series $\sum_{n=0}^{\infty}\frac{(\mathrm{n}\,!)^k}{(\mathrm{kn})\,!}\mathrm{x}^n$ is

(a.) k

(b.) k^{-k}

(c.) k^k

(d.) ∞

3. Let $\{a_n : n \geq 1\}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} |a_n|$ is divergent. Let R be the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$. then we can conclude that

(a.) $0 < R < 1$

(b.) $R = 1$

(c.) $1 < R < \infty$

(d.) $R = \infty$

4. Let f be a monotonically increasing function from $[0,1]$ into $[0,1]$. Which of the following statements is/are true?

(a.) f must be continuous at all but finitely many points in $[0,1]$

(b.) f must be continuous at all but countably many points in $[0,1]$

(c.) f must be Riemann integrable

(d.) f must be Lebesgue integrable

5. Let α, p be real numbers and $\alpha > 1$

(a.) If $p > 1$ then $\int_{-\infty}^{\infty}\frac{1}{|x|^{p\alpha}}dx < \infty$

(b.) If $p > \frac{1}{\alpha}$ then $\int_{-\infty}^{\infty}\frac{1}{|x|^{p\alpha}}dx < \infty$

(c.) If $p < \frac{1}{\alpha}$ then $\int_{-\infty}^{\infty}\frac{1}{|x|^{p\alpha}}dx < \infty$

(d.) For any $p \in R$ we have $\int_{-\infty}^{\infty}\frac{1}{|x|^{p\alpha}}dx = \infty$

6. Let $L = \int_0^1 \frac{dx}{1+x^8}$. Then

(a.) $L < 1$

(b.) $L > 1$

(c.) $L < \frac{\pi}{4}$

(d.) $L > \frac{\pi}{4}$

7. Let, f, g and h be bounded functions on the closed interval $[a, b]$, such that $f(x) \leq g(x) \leq h(x)$ for all $x \in [a, b]$. Let $P = \{a = a_0 < a_1 < a_2 < \ldots < a_n = b\}$ be a partition of $[a, b]$. We denote by $U(f, P)$ and $L(f, P)$, the upper and lower Riemann sums of f with respect to the partition P and similarly for g and h. Which of the following statements is necessarily true?

(a.) If $U(h, P) - U(f, P) < 1$ then $U(g, P) - L(g, P) < 1$

(b.) If $L(h, P) - L(f, P) < 1$ then $U(g, P) - L(g, P) < 1$

(c.) If $U(h, P) - L(f, P) < 1$ then $U(g, P) - L(g, P) < 1$

(d.) If $L(h, P) - U(f, P) < 1$ then $U(g, P) - L(g, P) < 1$

8. The power series $\sum_{n=0}^{\infty}\frac{\left[2+(-1)^n\right]^n}{3^n}x^n$ converges

(a.) Only for $x=0$

(b.) For all $x\in\mathbb{R}$

(c.) Only for $-1<x<1$

(d.) Only for $-1<x\le 1$

9. Area enclosed between x-axis from a to b and the curve $f(x)$ is finite when

(a.) $a=0,\ b=\infty,\ f(x)=e^{-5x^5}$

(b.) $a=-\infty,\ b=\infty,\ f(x)=e^{-5x^5}$

(c.) $a=-7,\ b=\infty,\ f(x)=\frac{1}{x^4}$

(d.) $a=-7,\ b=7,\ f(x)=\frac{1}{x^4}$

10. Let f be a continuously differentiable real-valued function on $[a,\ b]$ such that $|f'(x)|\le K$ for all $x\in[a,\ b]$. For a partition $P=\{a=a_0<a_1<\ldots<a_n=b\}$, let $U(f,\ P)$ and $L(f,\ P)$ denote the upper and lower Riemann sums of f with respect to P. Then

(a.) $|L(f,\ P)|\le K(b-a)\le|U(f,\ P)|$

(b.) $U(f,\ P)-L(f,\ P)\le K(b-a)$

(c.) $U(f,P)-L(f,P)\le K\|P\|$, where $\|P\|=\max\limits_{0\le i\le n-1}(a_{i+1}-a_i)$ is the norm of the partition

(d.) $U(f,\ P)-L(f,\ P)\le K\|P\|(b-a)$

11. Which of the conditions below imply that a function $f:[0,\ 1]\to\mathbb{R}$ is necessarily of bounded variation?

(a.) f is a monotone function on $[0,\ 1]$

(b.) f is a continuous and monotone function on $[0,\ 1]$

(c.) f has a derivative at each $x\in(0,\ 1)$

(d.) f has a bounded derivative on the interval $(0,\ 1)$.

12. Define $f:[0,\ 1]\to[0,\ 1]$ by $f(x)=\frac{2^k-1}{2^k}$ for $f\in\left[\frac{2^{k-1}-1}{2^{k-1}},\frac{2^k-1}{2^k}\right], k\ge 1$. Then f is a Riemann-integrable function such that

(a.) $\int_0^1 f(x)\,dx=\frac{2}{3}$

(b.) $\frac{1}{2}<\int_0^1 f(x)\,dx<\frac{2}{3}$

(c.) $\int_0^1 f(x)\,dx=1$

(d.) $\frac{2}{3}<\int_0^1 f(x)\,dx<1$

13. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $\int\limits_0^\infty f(x)\,dx$ exists. Which of the following statements are correct?

(a.) If $\lim\limits_{x\to\infty}f(x)$ exists, then $\lim\limits_{x\to\infty}f(x)=0$

(b.) The limit $\lim\limits_{x\to\infty}f(x)$ must exist and is zero

(c.) In case f is a non-negative function, $\lim\limits_{x\to\infty}f(x)$ must exist and is zero

(d.) In case f is a differentiable function $\lim\limits_{x\to\infty}f'(x)$ must exist and is zero

14. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function such that f' is bounded. Given a closed and bounded interval $[a,\ b]$, and a partition $P=\{a_0=a<a_1<\ldots<a_n=b\}$ of $[a,\ b]$, let $M(f,\ P)$ and $m(f,\ P)$ denote, respectively, the upper Riemann sum and the lower Riemann sum of f with respect to P. Then

(a.) $\left|M(f,P)-\int\limits_a^b f(x)\,dx\right|\le(b-a)$

$\sup\{|f(x)|:x\in[a,b]\}$

(b.) $\left|m(f,P)-\int\limits_a^b f(x)\,dx\right|\le(b-a)$

$\inf\{|f(x)|:x\in[a,b]\}$

(c.) $\left|M(f,P)-\int_a^b f(x)\,dx\right| \le (b-a)^2$

$\sup\{|f'(x)| : x\in[a,b]\}$

(d.) $\left|m(f,P)-\int_a^b f(x)\,dx\right| \le (b-a)^2$

$\inf\{|f'(x)| : x\in[a,b]\}$

15. Let $f:[0,1]\to\mathbb{R}$ be continuous such that $f(t)\ge 0$ for all t in $[0,1]$. Define $g(x)=\int_0^x f(t)\,dt$ then

(a.) g is monotone and bounded

(b.) g is monotone, but not bounded

(c.) g is bounded, but not monotone

(d.) g is neither monotone nor bounded

16. Let f be a continuous function on $[0,1]$ with $f(0)=1$. Let $G(a)=\frac{1}{a}\int_0^a f(x)\,dx$

(a.) $\lim_{a\to 0} G(a)=\frac{1}{2}$

(b.) $\lim_{a\to 0} G(a)=1$

(c.) $\lim_{a\to 0} G(a)=0$

(d.) The limit $\lim_{a\to 0} G(a)$ does not exist

17. Match the following

A. $\int_1^2 \frac{x}{\sqrt{x-1}}dx$ B. $\int_0^{\pi} \frac{dx}{1+\cos x}$

(i) Divergent & diverges to $+\infty$

(ii) Convergent & value is $\frac{4}{3}$

(iii) Divergent & diverges to $-\infty$

(iv) Convergent & value is $\frac{8}{3}$

(a.) A-1, B-2
(b.) A-3, B-4
(c.) A-4, B-1
(d.) A-2, B-3

18. Suppose f is an increasing real valued function on $[0,\infty)$ with $f(x)>0\ \forall x$ and let $g(x)=\frac{1}{x}\int_0^x f(u)\,du;\ 0<x<\infty$. Then which of the following are true:

(a.) $g(x)\le f(x)$ for all $x\in(0,\infty)$

(b.) $xg(x)\le f(x)$ for all $x\in(0,\infty)$

(c.) $xg(x)\ge f(0)\ x\in(0,\infty)$

(d.) $yg(y)-xg(x)\le(y-x)f(y)$ for all $x<y$

19. Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases} x\cos(\pi/(2x)) & \text{if } x\ne 0,\\ 0 & \text{if } x=0\end{cases}$ then

(a.) f is continuous on $[0,1]$

(b.) f is of bounded variation on $[0,1]$

(c.) f is differentiable $[0,1]$ and its derivative f' is bounded on $(0,1)$

(d.) f is Riemann integrable on $[0,1]$

20. Let $f:[0,\infty)\to\mathbb{R}$, and $g:[0,\infty)\to\mathbb{R}$ be continuous functions satisfying $\int_0^{f(x)} t^2\,dt = x^3(1+x)^2$ and $\int_0^{x^2(1+x)} g(t)\,dt = x$ for all $x\in[0,\infty)$. Then $f(2)+g(2)$ is equal to

(a.) 0

(b.) 5

(c.) 6

(d.) 11

21. Find the value of p for which the integral $\int_0^1 \frac{x^{p-1}}{1+x}dx$ is convergent.

(a.) Convergent if $p>1$

(b.) Convergent if $p=1$

(c.) Convergent if $p=0$

(d.) Convergent iff $p>0$

22. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Which of the following is always true?

(a.) $f^{-1}(U)$ is open for all open sets $U\subseteq\mathbb{R}$

(b.) $f^{-1}(C)$ is closed for all closed sets $C\subseteq\mathbb{R}$

(c.) $f^{-1}(K)$ is compact for all compact sets $K\subseteq\mathbb{R}$

(d.) $f^{-1}(G)$ is connected for all connected sets $G\subseteq\mathbb{R}$

23. Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable with $0<f'(x)<1$ for all x. Then

(a.) f is increasing and f is bounded.

(b.) f is increasing and f is Riemann integrable on i.

(c.) f is increasing and f is uniformly continuous.

(d.) f is of bounded variation.

24. c is a fixed point in $[0,\ 1]$. A function $f:[0,\ 1]\to\mathbb{R}$ is defined by $f(x)=\begin{cases} c & \text{if } 0\le x\le c \\ 2c & \text{if } c<x\le 1\end{cases}$. It is given that the Riemann integral $\int_0^1 f(x)dx$ is $\frac{7}{16}$. Then the value of c is

(a.) $\frac{1}{2}$

(b.) $\frac{1}{3}$

(c.) $\frac{1}{4}$

(d.) $\frac{1}{5}$

25. The improper integral $\int_1^\infty \frac{x}{1+x^2}\sin x dx$ is

(a.) Divergent

(b.) Convergent

(c.) Oscillatory

(d.) Diverges to $-\infty$

26. Let $f(x)=x^3-\sin x,\ x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Then f is

(a.) Bounded and of bounded variation

(b.) Unbounded and of bounded variation

(c.) Unbounded but not of bounded variation

(d.) Bounded but not of bounded variation

27. If $f:[1,\ 2]\to\mathbf{R}$ is a non-negative Riemann-integrable function such that $\int_1^2 \frac{f(x)}{\sqrt{x}}dx=k\int_1^2 f(x)dx\ne 0$, then k belongs to the interval

(a.) $\left[0,\frac{1}{3}\right]$

(b.) $\left(\frac{1}{3},\frac{2}{3}\right]$

(c.) $\left(\frac{2}{3},1\right]$

(d.) $\left(1,\frac{4}{3}\right]$

28. The improper integral $\int_0^4 \frac{dx}{x(4-x)}$ is

(a.) Convergent & value in 6

(b.) Divergent & diverges to $+\infty$

(c.) Convergent & value is 4

(a.) None of these

29. Consider the power series $\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$ and $\sum_{n=1}^{\infty}\frac{x^n}{n}$. Then

(a.) both converge on $(-1,1]$

(b.) both converge on $[-1,1)$

(c.) exactly one of them converges on $(-1,1]$

(d.) none of them converges on $[-1,1)$

30. Which one of the following does NOT hold for all continuous functions $f:[-\pi,\pi]\to\mathbb{R}$?

(a.) If $f(-t)=f(t)$ for each $t\in[-\pi,\pi]$, then $\int_{-\pi}^{\pi} f(t)\,dt=2\int_{0}^{\pi} f(t)\,dt$

(b.) If $f(-t)=-f(t)$ for each $t\in[-\pi,\pi]$, then $\int_{-\pi}^{\pi} f(t)\,dt=0$

(c.) $\int_{-\pi}^{\pi} f(t)\,dt=-\int_{-\pi}^{\pi} f(t)\,dt$

(d.) There is an α with $-\pi<\alpha<\pi$ such that $\int_{-\pi}^{\pi} f(t)\,dt=2\pi f(\alpha)$

31. For a positive real number p, let (f_n: n = 1, 2,....) be a sequence of functions defined on [0, 1] by $f_n(x)=\begin{cases} n^{p+1}x, & \text{if } 0\le x<\frac{1}{n} \\ \frac{1}{x^p}, & \text{if } \frac{1}{n}\le x\le 1 \end{cases}$ Let $f(x)=\lim_{n\to\infty} f_n(x), x\in[0,1]$. Then, on $[0,1]$

(a.) f is Riemann integrable

(b.) the improper integral $\int_0^1 f(x)\,dx$ converges for p ≥ 1

(c.) the improper integral $\int_0^1 f(x)\,dx$ converges for p < 1

(d.) f_n converges uniformly

32. Let $\chi_A(x)$ denote the function which is 1 if $x\in A$ and 0 otherwise. Consider

$$f(x)=\sum_{n=1}^{200}\frac{1}{n^6}\chi_{\left[0,\frac{n}{200}\right]}(x), x\in[0,1]$$

Then $f(x)$ is

(a.) Riemann integrable on $[0,1]$

(b.) Lebesgue integrable on $[0,1]$

(c.) Is a continuous function on $[0,1]$

(d.) Is a monotone function on $[0,1]$

33. Define f on $[0,1]$.

$f(x)=\begin{cases} x^2 \text{ if x is rational} \\ x^3 \text{ if x is irrational} \end{cases}$. Then

(a.) f is not Riemann integrable on $[0,1]$.

(b.) f is Riemann integrable and $\int_0^1 f(x)\,dx=\frac{1}{4}$.

(c.) f is Riemann integrable and $\int_0^1 f(x)\,dx=\frac{1}{3}$.

(d.) $\frac{1}{4}=\underline{\int_0^1} f(x)\,dx<\overline{\int_0^1} f(x)\,dx=\frac{1}{3}$, where $\underline{\int_0^1} f(x)\,dx$ and $\overline{\int_0^1} f(x)\,dx$ are the lower and upper Riemann integrals of f.

34. Consider the improper Riemann integral $\int_0^x y^{-1/2}\,dy$.

This integral is:

(a.) Continuous in $[0,\infty)$.

(b.) continuous only in $(0,\infty)$.

(c.) discontinuous in $(0,\infty)$.

(d.) discontinuous only in $(1/2,\infty)$.

35. Let $f:[0,1]\to\mathbb{R}$ and $g:[0,1]\to\mathbb{R}$ be two functions defined by

$f(x)=\begin{cases} \frac{1}{n} & \text{If } x=\frac{1}{n}, n\in\mathbb{N} \\ 0 & \text{Otherwise} \end{cases}$ and

$g(x)=\begin{cases} n & \text{If } x=\frac{1}{n}, n\in\mathbb{N} \\ 0 & \text{Otherwise} \end{cases}$

(a.) Both f and g are Riemann integrable

(b.) f is Riemann integrable but g is not

(c.) g is Riemann integrable but f is not

(d.) Neither f nor g is Riemann integrable

ASSIGNMENT SHEET - 9

1. For $n \geq 1$, let $g_n(x) = \sin^2\left(x + \frac{1}{n}\right), x \in [0, \infty)$ and $f_n(x) = \int_0^x g_n(t)\,dt$. Then

(a.) $\{f_n\}$ converges pointwise to a function f on $[0, \infty)$, but does not converge uniformly on $[0, \infty)$.

(b.) $\{f_n\}$ does not converge to any function on $[0, \infty)$.

(c.) $\{f_n\}$ converges uniformly on $[0, 1)$.

(d.) $\{f_n\}$ converges uniformly on $[0, \infty)$.

2. Let $\{b_n\}$ and $\{c_n\}$ be sequence of real numbers then a necessary and sufficient condition for the sequence of polynomials $f_n(x) = b_n x + c_n x^2$ to converge uniformly to 0 on the real line is

(a.) $\lim\limits_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 0$

(b.) $\sum_{n=1}^{\infty} |b_n| < \infty$ and $\sum_{n=1}^{\infty} |c_n| < \infty$

(c.) There exists a positive integer N such that $b_n = 0$ and $c_n = 0$ for all $n > N$

(d.) $\lim_{n\to\infty} c_n = 0$

3. Let $\{f_n\}$ be a sequence of continuous functions on R.

(a.) If $\{f_n\}$ converges to f pointwise on R then $\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$

(b.) If $\{f_n\}$ converges to f uniformly on R then $\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$

(c.) If $\{f_n\}$ converges to f uniformly on R then f is continuous on R

(d.) There exists a sequence of continuous functions $\{f_n\}$ on R, such that $\{f_n\}$ converges to f uniformly on R, but $\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx \neq \int_{-\infty}^{\infty} f(x)\,dx$

4. Let $P_n(x) = a_n x^2 + b_n x + c_n$ be a sequence of quadratic polynomials where $a_n, b_n, c_n \in R$ for all $n \geq 1$. Let $\lambda_0, \lambda_1, \lambda_2$ be distinct real numbers such that $\lim\limits_{n\to\infty} P_n(\lambda_0) = A_0$, $\lim\limits_{n\to\infty} P_n(\lambda_1) = A_1$ and $\lim\limits_{n\to\infty} P_n(\lambda_2) = A_2$. Then

(a.) $\lim\limits_{n\to\infty} P_n(x)$ exists for all $x \in R$

(b.) $\lim\limits_{n\to\infty} P_n'(x)$ exists for all $x \in R$

(c.) $\lim\limits_{n\to\infty} P_n\left(\frac{\lambda_0, \lambda_1, \lambda_2}{3}\right)$ does not exist

(d.) $\lim\limits_{n\to\infty} P_n'\left(\frac{\lambda_0, \lambda_1, \lambda_2}{3}\right)$ does not exist

5. Let $f_n : [1, 2] \to [0, 1]$ be given by $f_n(x) = (2-x)^n$ for all non-negative integers n. Let $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for $1 \leq x \leq 2$. Then which of the following is true?

(a.) f is a continuous function on $[1, 2]$.

(b.) f_n converges uniformly to f on $[1, 2]$ as $n \to \infty$.

(c.) $\lim_{n\to\infty} \int_1^2 f_n(x)\,dx = \int_1^2 f(x)\,dx$.

(d.) for any $a \in (1, 2)$ we have $\lim\limits_{n\to\infty} f'_n(a) \neq f'(a)$.

6. Let $f_n(x) = \begin{cases} 1 - nx & \text{for } x \in \left[0, \frac{1}{n}\right] \\ 0 & \text{for } x \in \left[\frac{1}{n}, 1\right] \end{cases}$ Then

(a.) $\lim\limits_{n\to\infty} f_n(x)$ defines a continuous function on $[0, 1]$

(b.) $\{f_n\}$ converges uniformly on $[0, 1]$

(c.) $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in [0, 1]$

(d.) $\lim\limits_{n\to\infty} f_n(x)$ exists for all $x \in [0, 1]$

7. Let $f_n(x)=x^{1/n}$ for $n\in[0,\ 1]$. Then

(a.) $\lim\limits_{n\to\infty} f_n(x)$ exists for all $x\in[0,\ 1]$

(b.) $\lim\limits_{n\to\infty} f_n(x)$ defines a continuous function on $[0,\ 1]$

(c.) $\{f_n\}$ converges uniformly on $[0,\ 1]$

(d.) $\lim\limits_{n\to\infty} f_n(x)=0$ for all $x\in[0,\ 1]$.

8. Let $f(x)=\sum\limits_{n=1}^{\infty}\dfrac{\sin(nx)}{n^2}$. Then

(a.) $\lim\limits_{x\to 0} f(x)=0$

(b.) $\lim\limits_{x\to 0} f(x)=1$

(c.) $\lim\limits_{x\to 0} f(x)=\dfrac{\pi^2}{6}$

(d.) $\lim\limits_{x\to 0} f(x)$ does not exist

9. The series $\sum\limits_{m=1}^{\infty} x^{\ln m}$, $x>0$, is convergent on the interval

(a.) $(0,\ 1/e)$

(b.) $(1/e,\ e)$

(c.) $(0,\ e)$

(d.) $(1,\ e)$

10. Let $f_n(x)=\dfrac{x}{\{(n-1)x+1\}\{nx+1\}}$ and $s_n(x)=\sum\limits_{j=1}^{n} f_j(x)$ for $x\in[0,1]$. The sequence $\{s_n\}$

(a.) Converges uniformly on $[0,1]$

(b.) Converges pointwise on $[0,1]$ but not uniformly

(c.) Converges pointwise for $x=0$ but not for $x\in(0,1]$

(d.) Does not converge for $x\in[0,1]$

11. Let $\{f_n\}$ be a sequence of real valued differentiable function on $[a,b]$ such that $f_n(x)\to f(x)$ as $n\to\infty$ for every $x\in[a,b]$ and for some Riemann-integrable function $f:[a,b]\to\mathbf{R}$. Consider the statements:

$P_1:\{f_n\}$ converges uniformly

$P_2:\{f_n'\}$ converges uniformly

$P_3:\int_a^b f_n(x)\,dx\to\int_a^b f(x)\,dx$

$P_4: f$ is differentiable

Then which one of the following need NOT be true

(a.) P_1 implies P_3

(b.) P_2 implies P_1

(c.) P_2 implies P_4

(d.) P_3 implies P_1

12. Let $f_n(x)=\dfrac{x^n}{1+x}$ and $g_n(x)=\dfrac{x^n}{1+nx}$ for $x\in\mathbf{N}$. Then on the interval $[0,1]$,

(a.) both $\{f_n\}$ and $\{g_n\}$ converges uniformly

(b.) neither $\{f_n\}$ nor $\{g_n\}$ converges uniformly

(c.) $\{f_n\}$ converges uniformly but $\{g_n\}$ does not converge uniformly

(d.) $\{g_n\}$ converges uniformly but $\{f_n\}$ does not converges uniformly

13. Which of the following sequence $\{f_n\}_{n=1}^{\infty}$ of functions does NOT converge uniformly on [0, 1]?

(a.) $f_n(x)=\dfrac{e^{-x}}{n}$

(b.) $f_n(x)=(1-x)^n$

(c.) $f_n(x)=\dfrac{x^2+nx}{n}$

(d.) $f_n(x)=\dfrac{\sin(nx+n)}{n}$

14. Which one of the following statements holds?

(a.) The series $\sum\limits_{n=0}^{\infty} x^n$ converges for each $x\in[-1,1]$

(b.) The series $\sum\limits_{n=0}^{\infty} x^n$ converges uniformly in $(-1,1)$

(c.) The series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges for each $x \in [-1,1]$

(d.) The series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges uniformly in $(-1,1)$

15. Consider two sequences $\{f_n\}$ and $\{g_n\}$ of functions where $f_n : [0,1] \to R$ and $g_n : R \to R$ are defined by $f_n(x) = x^n$ and

$$g_n(x) = \begin{cases} \cos(x-n)\pi/2 & \text{If } x \in [n-1, n+1] \\ 0 & \text{otherwise} \end{cases}$$

Then1

(a.) Neither $\{f_n\}$ nor $\{g_n\}$ is uniformly convergent

(b.) $\{f_n\}$ is not uniformly convergent but $\{g_n\}$ is

(c.) $\{g_n\}$ is not uniformly convergent but $\{f_n\}$

(d.) Both $\{f_n\}$ and $\{g_n\}$ are uniformly convergent

16. Let $f_x(x) = \frac{\sin x}{\sqrt{n}}, n = 1, 2, \ldots$ and $x \in [-1,1]$. Choose incorrect

(a.) $\{f_n(x)\}_{n=1}^{\infty}$ does not converge uniformly in $[-1,1]$

(b.) $\lim_{n\to\infty} \int_{-1}^{1} f_n(n)\,dx \neq 0$

(c.) $\{f_n'(x)\}$ does not converge uniformly in $[-1,1]$

(d.) $f_n(x), n = 1, 2$ is not uniformly continuous in $[-1,1]$

17. Find out which of the following series converge uniformly for $x \in (-\pi, \pi)$.

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(a.) $\sum_{n=1}^{\infty} \frac{e^{-n|x|}}{n^3}$

(b.) $\sum_{n=1}^{\infty} \frac{\sin(xn)}{n^5}$

(c.) $\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n$

(d.) $\sum_{n=1}^{\infty} \frac{1}{\left((x+\pi)n\right)^2}$

18. Let $f_n(x) = (-x)^n, x \in [0,1]$. Then decide which of the following are true.

(a.) there exist a pointwise convergent subsequence of f_n

(b.) f_n has non pointwise convergent subsequence.

(c.) f_n converges pointwise everywhere.

(d.) f_n has exactly one pointwise convergent subsequence.

19. Consider all sequences $\{f_n\}$ of real valued continuous functions on $[0, \infty)$. Identify which of the following statements are correct.

(a.) If $\{f_n\}$ converges to f pointwise on $[0, \infty)$, then $\lim_{n\to\infty} \int_0^{\infty} f_n(x)\,dx = \int_0^{\infty} f(x)\,dx$

(b.) If $\{f_n\}$ converges to f uniformly on $[0, \infty)$, then $\lim_{n\to\infty} \int_0^{\infty} f_n(x)\,dx = \int_0^{\infty} f(x)\,dx$

(c.) If $\{f_n\}$ converges to f uniformly on $[0, \infty)$, then f is continuous on $[0, \infty)$.

(d.) There exists a sequence of continuous functions $\{f_n\}$ on $[0, \infty)$ such that $\{f_n\}$ converges to f uniformly on $[0, \infty)$ but $\lim_{n\to\infty} \int_0^{\infty} f_n(x)\,dx \neq \int_0^{\infty} f(x)\,dx$

20. Which one of the following statements is true for the sequence of functions

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$$f_n(x) = \frac{1}{n^2 + x^2}, n = 1, 2, \cdots, x \in [1/2, 1]?$$

(a.) The sequence is monotonic and has 0 as the limit for all $x \in [1/2, 1]$ as $n \to \infty$

(b.) The sequence is not monotonic but has $f(x) = \frac{1}{x^2}$ as the limit as $n \to \infty$.

(c.) The sequence is monotonic and has $f(x) = \frac{1}{x^2}$ as the limit as $n \to \infty$.

(d.) The sequence is not monotonic but has 0 as the limit.

ASSIGNMENT SHEET - 10

1. Let $A=\{(x,y)\in\mathbb{R}^2 : x+y\neq -1\}$ Define $f:A\to\mathbb{R}^2$ by $f(x,y)=\left(\frac{x}{1+x+y},\frac{y}{1+x+y}\right)$. Then,

(a.) The Jacobian matrix of f does not vanish on A

(b.) f is infinitely differentiable on A

(c.) f is injective on A

(d.) $f(A)=\mathbb{R}^2$

2. Define $f:\mathbb{R}^2\to\mathbb{R}^2$ by $f(x,y)=(x+2y+y^2+|xy|, 2x+y+x^2+|xy|$ for $(x,y)\in\mathbb{R}^2$. Then

(a.) f is discontinuous at $(0,0)$

(b.) f is continuous at $(0,0)$ but not differentiable at $(0,0)$

(c.) f is differentiable at $(0,0)$

(d.) f is differentiable at $(0,0)$ and the derivative $Df(0,0)$ is invertible

3. Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be given by $f(x,y)=(x+y, xy)$. Then

(a.) f is not differentiable at the point $(0,0)$.

(b.) The derivative of f is invertible except on the set $\{(x,y)\in\mathbb{R}^2 \mid x=y\}$.

(c.) The inverse image of each point in $\mathbb{R}^2$ under f has at most two elements.

(d.) f is surjective.

4. Let $f:\mathbb{R}^m\to\mathbb{R}^m$ be a differential function. Let $Df(x)$ be the derivative of f at $x\in\mathbb{R}^m$. Which of the following is/are correct?

(a.) $Df(x)(u)=0$ for all u in $\mathbb{R}^m$.

(b.) $Df(x)(u)=0$ for all u in $\mathbb{R}^m$ and some $x\in\mathbb{R}^m$ only if f is a constant.

(c.) $Df(x)(u)=0$ for all $u\in\mathbb{R}^m$ and all $x\in\mathbb{R}^m$ only if f is a constant.

(d.) If f is not a constant function, then $Df(x)$ is a one to one function for some $x\in\mathbb{R}^m$

5. If $f:S\to S$ is a function, then we denote by f^k, the function $f\circ f\circ\cdots\circ f$ (k times). Let f_1 and f_2 be two functions defined on $\mathbb{R}^2$ as follows

$f_1(x,y)=(x+1,y+3)$, $f_2(x,y)=(x-3,y-2)$. Then

(a.) For any positive integer k, there exists a unique $(a,b)\in\mathbb{R}^2$, such that $f_1^k(0,0)=f_2^k(a,b)$.

(b.) For any real number a and any positive integer, there is at most one solution y for $f_1^k(0,0)=f_2^k(a,y)$

(c.) There exists $(a,b)\in\mathbb{R}^2$ such that $f_1^k(a,b)\neq f_2^k(x,y)$ for any $(x,y)\in\mathbb{R}^2$ and any positive integer k.

f_1 is linear transformation.

6. Let $f:\mathbb{R}^n\to\mathbb{R}$ be the map $f(x_1,\dots,x_n)=a_1x_1+\dots+a_nx_n$, where $a=(a_1,\dots,a_n)$ is a fixed non-zero vector. Let $Df(0)$ denote the derivative of f at 0. Which of the following are correct?

(a.) $(Df)(0)$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$

(b.) $[(Df)(0)](a)=\|a\|^2$

(c.) $[(Df)(0)](a)=0$

(d.) $[(Df)(0)](b)=a_1b_1+\dots+a_nb_n$ for $b=(b_1,\dots,b_n)$

7. Let $f(x,y)=\sqrt{|xy|}$. Then

(a.) f_x and f_y do not exist at $(0,0)$

(b.) $f_x(0,0)=1$

(c.) $f_y(0,0)=0$

(d.) f is differentiable at $(0,0)$

8. Let $z=x+iy$ and $f:\mathbb{R}^2\to\mathbb{R}^2$ be the function $f(x,y)=f(z)=z^2=(x^2-y^2, 2xy)\in\mathbb{R}^2$. Let $(Df)(a)$ denote the derivative of f at a. Which of the following are true?

(a.) $(Df)(a)h=2\,a\,h$, where $a=a_1+ia_2$ and $h=h_1+ih_2$

(b.) $(Df)(a)=2\begin{pmatrix}a_1 & -a_2\\ a_2 & a_1\end{pmatrix}, a=(a_1,a_2)\in\mathbb{R}^2$

(c.) f is one to one on $\mathbb{R}^2$

(d.) For any $a\in\mathbb{R}^2/\{(0,0)\}$, f is one to one on some neighbourhood of a.

9. Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be the function defined by $f(x)=x\|x\|^2$ for $x\in\mathbb{R}^n$. Which of the following statements are correct?

(a.) $(Df)(0)=0$

(b.) $(Df)(x)=0$ for all $x\in\mathbb{R}^n$

(c.) f is one-one

(d.) f has an inverse

10. Let $f:A\cup E\to\mathbb{R}^2$ be differentiable, where $A=\left\{(x,y)\in\mathbb{R}^2:\frac{1}{2}<x^2+y^2<1\right\}$ and $E=\left\{(x,y)\in\mathbb{R}^2:(x-2)^2+(y-2)^2<\frac{1}{2}\right\}$. Let Df be the derivative of the function f. Which of the following are necessarily correct?

(a.) If $(Df)(x,y)=0$ for all $(x,y)\in A\cup E$, then f is constant

(b.) If $(Df)(x,y)=0$ for all $(x,y)\in A$, then f is constant on A

(c.) If $(Df)(x,y)=0$ for all $(x,y)\in E$, then f is constant on E

(d.) If $(Df)(x,y)=0$ for all $(x,y)\in A\cup E$, then for some $(x_0,y_0),(x_1,y_1)\in\mathbb{R}^2$, $f(x,y)=(x_0,y_0)$ for all $(x,y)\in A$ and $f(x,y)=(x_1,y_1)$ for all $(x,y)\in E$.

11. Let $L:\mathbb{R}^n\to\mathbb{R}$ be the function $L(x)=\langle x,\ y\rangle$, where $\langle\cdot,\cdot\rangle$ is some inner product on $\mathbb{R}^n$ and y is a fixed vector in $\mathbb{R}^n$. Further denote by DL, the derivative of L. Which of the following are necessarily correct?

(a.) $DL(u)=DL(v)$ for all $u,v\in\mathbb{R}^n$

(b.) $DL(0,0,...,0)=L$

(c.) $DL(x)=\|x\|^2$ for all $x\in\mathbb{R}^n$

(d.) $DL(1,1,...,1)=0$.

12. Let $f:[\pi,\ 2\pi]\to\mathbb{R}^2$ be the function $f(t)=(\cos t,\sin t)$. Which of the following are necessarily correct?

(a.) There exists $t_0\in[\pi,2\pi]$ such that $f'(t_0)=\frac{1}{\pi}(f(2\pi)-f(\pi))$

(b.) There does not exist any $t_0\in[\pi,2\pi]$ such that $f'(t_0)=\frac{1}{\pi}(f(2\pi)-f(\pi))$

(c.) There exists $t_0\in[\pi,2\pi]$ such that $\|f(2\pi)-f(\pi)\|\le\pi\|f'(t_0)\|$

(d.) $f'(t)=(-\sin t,\cos t)$ for all $t\in[\pi,2\pi]$.

13. Consider the map $f:\mathbb{R}^2\to\mathbb{R}^2$ defined by $f(x,y)=(7x+x^4,3x+4y+y^4)$. Then,

(a.) f is discontinuous at (0, 0).

(b.) f is continuous at (0, 0) and all directional derivatives exist at (0, 0)

(c.) f is differentiable at (0, 0) but the derivative Df (0, 0) is not invertible.

(d.) f is differentiable at (0, 0) and the derivative Df (0, 0) is invertible.

14. The map $L:\mathbb{R}^2\to\mathbb{R}^2$ given by $L(x,y)=(x,-y)$ is

(a.) Differentiable everywhere on $\mathbb{R}^2$

(b.) Differentiable only at $(0,\ 0)$

(c.) $DL(0,\ 0)=L$

(d.) $DL(x,\ y)=L$ for all $(x,y)\in\mathbb{R}^2$

15. Consider the map $f:\mathbb{R}^2\to\mathbb{R}^2$ defined by $f(x,y)=(3x-2y+x^2,4x+5y+y^2)$. Then

(a.) f is discontinuous at (0, 0)

(b.) f is continuous at (0,0) and all directional derivatives exists at (0, 0)

(c.) f is differentiable at (0,0) but the derivative $Df(0,\ 0)$ is not invertible

(d.) f is differentiable at (0,0) and the derivative $Df(0,\ 0)$ is invertible.

16. Let $f:\mathbb{R}^2 \to \mathbb{R}$ and $g:\mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y)=|x|+|y|$ and $g(x,\ y)=|xy|$. Then

(a.) f is differentiable at $(0,\ 0)$, but g is not differentiable at $(0,\ 0)$

(b.) g is differentiable at $(0,\ 0)$, but f is not differentiable at $(0,\ 0)$

(c.) Both f and g are differentiable at $(0,\ 0)$

(d.) Both f and g are continuous at $(0,\ 0)$

17. Let

$$f(x,y)=\begin{cases}\dfrac{xy}{\left(x^2+y^2\right)^{5/2}}\left[1-\cos\left(x^2+y^2\right)\right], & (x,y)\neq(0,0)\\ k & ,\ (x,y)=(0,0)\end{cases}$$

Then the value of k for which $f(x, y)$ is continuous at (0, 0) is

(a.) 0

(b.) $\frac{1}{2}$

(c.) 1

(d.) $\frac{3}{2}$

18. For $(x,y)\in R^2$ $f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2} & if\,(x,y)\neq 0\\ 0 & if\,(x,y)=0\end{cases}$

Then

(a.) f_x and f_y exist at $(0,0)$ f is continuous at $(0,0)$

(b.) f_x and f_y exist at $(0,0)$ and f is discontinuous at $(0,0)$

(c.) f_x and f_y do not exist at $(0,0)$, f is continuous at $(0,0)$

(d.) f_x and f_y do not exist at $(0,0)$ and f is discontinuous at $(0,0)$

19. Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x,y)=\begin{cases}x^2+y^2 & \text{If x and y are rational}\\ 0 & \text{Otherwise}\end{cases}$$

Then

(a.) f is not continuous at (0,0)

(b.) f is continuous at (0,0) but not differentiable at (0,0)

(c.) f is differentiable only at (0,0)

(d.) f is differentiable every where

20. If $f(x,y)=\begin{cases}x^3/\left(x^2+y^2\right), & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$, then at $(0,0)$

(a.) f_x, f_y do not exist

(b.) f_x, f_y exist and are equal

(c.) The directional derivative exists along any straight line

(d.) f is differentiable

21. Let

$f(x,y)=\dfrac{xy}{x^2+y^2},(x,y)\neq(0,0);f(0,0)=0$.

Then

(a.) f is continuous at (0,0) and the partial derivatives $f_x.f_y$ exist at every point of $\mathbb{R}^2$

(b.) f is discontinuous at (0,0) and $f_{x'}$ f exist at every point of $\mathbb{R}^2$

(c.) f is discontinuous at (0,0) and $f_x.f$ exist only at (0,0)

(d.) None of the above

22. Consider the function $f(x,y)=\dfrac{x^2}{y^2},(x,y)\in[1/2,3/2]\times[1/2,3/2]$. The derivative of the function at $(1,1)$ along the direction $(1,1)$

(a.) 0

(b.) 1

(c.) 2

(d.) -2

23. Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ be the function $f(r,\theta)=(r\cos\theta, r\sin\theta)$. Then for which of the open subsets U of $\mathbb{R}^2$ given below, f restricted to U admits an inverse?

(a.) $U=\mathbb{R}^2$

(b.) $U=\left\{(x,y)\in\mathbb{R}^2 : x>0, y>0\right\}$

(c.) $U=\left\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\right\}$

(d.) $U=\left\{(x,y)\in\mathbb{R}^2 : x<-1, y<-1\right\}$

24. Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be given by $f(x,y)=(x^2,y^2+\sin x)$. Then the derivative of f at (x,y) is the linear transformation given by

(a.) $\begin{pmatrix} 2x & 0 \\ \cos x & 2y \end{pmatrix}$

(b.) $\begin{pmatrix} 2x & 0 \\ 2y & \cos x \end{pmatrix}$

(c.) $\begin{pmatrix} 2y & \cos x \\ 2x & 0 \end{pmatrix}$

(d.) $\begin{pmatrix} 2x & 2y \\ 0 & \cos x \end{pmatrix}$

25. A function $f:\mathbb{R}^2\to\mathbb{R}$ is defined by $f(x,y)=xy$. Let $v=(1,2)$ and $a=(a_1,a_2)$ be two elements of $\mathbb{R}^2$. The directional derivative of f in the direction of v at a is

(a.) a_1+2a_2

(b.) a_2+2a_1

(c.) $\frac{a_2}{2}+a_1$

(d.) $\frac{a_1}{2}+a_2$

26. A function $f(x,y)$ on $\mathbb{R}^2$ has the following partial derivatives

$$\frac{\partial f}{\partial x}(x,y)=x^2,\frac{\partial f}{\partial y}(x,y)=y^2$$

Then

(a.) f has directional derivatives in all direction everywhere.

(b.) f has a derivative at all points.

(c.) f has directional derivative only along the direction $(1,1)$ everywhere.

(d.) f does not have directional derivatives in any direction everywhere.